Nonholonomic Relativistic Diffusion and Exact Solutions for Stochastic Einstein Spaces

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Abstract. We develop an approach to the theory nonholonomic relativistic stochastic processes in curved spaces. The Itô and Stratonovich calculus are formulated for spaces with conventional horizontal (holonomic) and vertical (nonholonomic) splitting defined by nonlinear connection structures. Geometric models of relativistic diffusion theory are elaborated for nonholonomic (pseudo) Riemannian manifolds and phase velocity spaces. Applying the anholonomic deformation method, the field equations in Einstein gravity and various modifications are formally integrated in general forms, with generic off–diagonal metrics depending on some classes of generating and integration functions. Choosing random generating functions we can construct various classes of stochastic Einstein manifolds. We show how stochastic gravitational interactions with mixed holonomic/ nonholonomic and random variables can be modelled in explicit form and study their main geometric and stochastic properties. Finally, there are analyzed the conditions when non–random classical gravitational processes transform into stochastic ones and inversely.

PACS. 02.50.Ey Stochastic processes – 02.90.+p Other topics in mathematical methods in physics – 02.40.-k Geometry, differential geometry, and topology – 04.20.Jb Exact solutions

1 Introduction

In recent years stochastic methods and the theory of (relativistic) diffusion on Lorentz manifolds, and (for instance) Finsler/ supersymmetric generalizations, re–attracted considerable interest and applications in various directions of mathematics, gravity and modern cosmology and astrophysics. During the last decade a lot of papers have been devoted to such issues (there are many ways and approaches); for reviews and original results in physics and mathematics, see [2,3,4,5,6,8,11,12] and references therein.

In a series of our works [17,18,21], we proved that the gravitational field equations in various theories of gravity (general relativity and extra dimension models, noncommutative, nonsymmetric, Finsler etc generalizations) can be integrated in very general forms using the so–called anholonomic deformation/ frame method. The idea of the method is to use some distortion tensors completely defined by a given, in general, off–diagonal (which can not be diagonalized by coordinate transforms) metric structure and uniquely deforming the Levi–Civita connection into "auxiliary" linear connections which are also compatible and uniquely defined by the same metric. For such a connection and with respect to some classes of nonholonomic frames of reference, the system of Einstein equations decouple split into conventional sub–systems which can be integrated "step by step" in very general forms. Imposing additional nonholonomic constraints on integral varieties, we can generate exact solutions in the general relativity theory.

In explicit form, the type of exact solutions depend on the class of generating and integration functions and considered boundary/initial conditions, on prescribed smooth classes of coefficients and sources of equations and on existing spacetime local and global symmetries for a chosen topology. We have to involve additional physical arguments in order to select a "physically important" exact solution with deterministic or random properties. The bulk of exact solutions in gravity theories were constructed for smooth and/or singular classes of metrics and there were analyzed possible perturbative deformations by quasi–classical quantum contributions and fluctuations of classical and quantum matter sources.

In this work we study new classes of stochastic gravitational processes modelled as exact solutions of the Einstein equations (of vacuum type, with cosmological constant and/or with various types of stochastic sources). There are developed certain methods for the theory of stochastic calculus and relativistic diffusion on curved spaces following the formalism of moving frames with associated nonlinear connection structure and respective constructions from

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the geometry of nonholonomic manifolds. We prove that such new classes of stochastic solutions in gravity are derived by integrating in general form certain systems of PDE with separation of equations[1] and stochastic generating functions and by imposing corresponding nonholonomic (equivalently, anholonomic, or non–integrable) constraints on gravitational vacuum and non–vacuum interactions. [2] Our aim is also to formulate special and general relativistic nonholonomic diffusion models and apply such stochastic techniques in modern gravity theory.

With respect to exact sure and stochastic solutions of the Einstein equation, we have to address to the general **problem of transition to stochastic phase** of solutions for a system of nonlinear partial differential equations (PDE) on curved manifold/ bundle spaces is the following:

- *If and under what conditions can an existing "non–random" solution (depending on certain smooth classes of generating and integration functions) be further transformed into a stochastic solution of the same system of equations?*
- *What types of nonholonomic constraints, boundary/inicial conditions, and corresponding data for generating/ integration functions will transform a nonlinear dynamical/evolution classical, or quantum process, into a stochastic one, and inversely?*

In brief, this can be equivalently posed as a problem for formulating some criteria for generating nonlinearly stochastic processes (for certain well defined conditions, diffusion). In general, the solutions of systems of PDE can be with mixed sure and random variables.

We shall construct exact stochastic solutions of Einstein equations in explicit form for vacuum and non–vacuum gravitational configurations. Such spacetimes are described by generic off–diagonal metrics, the nonlinear gravitational interactions being subjected to certain types of nonholonomic (equivalently, anholonomic, or non–integrable) constraints/ conditions which may give rise, or not, to random gravitational processes. A cosmological constant, with possible local anisotropic polarizations, and matter sources present additional issues to be addressed. [3]

A partner work [1] contains developments of the results of this paper for stochastic Ricci–Finsler flows and gravitational interactions/ evolutions modeling porous media and self–organized criticality.

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[1] it should be not confused with the method of separation of variables

[2] For non–experts on probability and stochastic calculus (i.e. researches skilled in differential geometry and applications), we remember some main concepts and ideas which are important for definition of relativistic diffusion theory (see details in [22, 23, 24, 25]): Physicists have abandoned determinism as a fundamental description of reality and began developing probabilistic (that is, stochastic) models of natural phenomena long before quantum mechanics and physics. Classical uncertainty preceded quantum uncertainty (in our approach derived from solutions of nonlinear systems of equations). P. Langevin, was the first to apply Newton's second law to a "Brownian particle" on which the total force included a random components. The time evolution of a random variable is called a random or stochastic process. Thus $X(\tau)$ denotes a stochastic process. The time τ evolution of a sure variable is called a deterministic process and denoted $x(\tau)$.
Random and Sure Variables: A quantity that, under given conditions, can assume different values is a *random variable.* It matters not whether the random variation is intrinsic and unavoidable (for instance, a consequence of quantum effects) or an artifact of our ignorance. A random variable is conceptually distinct from a "certain" or "sure", i.e. deterministic, variable.
The *expected value* of a random variable X is a function that turns the probabilities $P(x)$ into a sure variable called the mean of X. The mean $\langle P \rangle$ parametrizes the random variable X, but also do all the *moments* $\langle X^n \rangle$ $(n = 0, 1, ...)$ and *moments about the mean* $\langle (X - \langle X \rangle)^n \rangle$. A continuous random variable $X(\tau)$ is completely defined by its probability density $p(x)$.
A *memoryless* process is a *Markov* process. The Wiener (Browninan) process, defined by the Markov propagator with a "simple" parameter ρ^2 is the simplest of all Markov processes. The Wiener process is, on its domain, everywhere continuous but nowhere smooth. This special property makes the Wiener process dynamical equation a different kind of mathematical object – a *stochastic differential equation* (SDE). It is characterized by a diffusion constant. For a stochastic process $X(t)$, the *mean function* is $\mu(t) = \mathbb{E}[X(t)]$ and the *covariance function is* $k(t, t') = \mathbb{E}[(X(t) - \mu(t))(X(t') - \mu(t'))]$. *Gaussian processes* are stochastic processes defined by their mean and covariance functions.
A diffusion equation is mathematically equivalent to a stochastic dynamical equation. In gravity theory, random variables are introduced when some components of metric are changed by stochastic forces (generating functions) driven by a Wiener process. Markov process can be considered but with "some" memory on hyperbolicity (finite speed of light), Lorentz transforms in special relativity, and Mach principle in general relativity, as well on Einstein equations etc. This is encoded in the definition of Stratonovich integral on curved/nonholonomic spaces.

[3] A nonholonomic manifold is defined by a pair $(\mathbf{V}, \mathcal{N})$, where $\mathbf{V}$ is a manifold and $\mathcal{N}$ is a non–integrable distribution. We have to involve in our research certain methods from the geometry of nonholonomic distributions with associated nonlinear connection structure (geometry of nonholonomic manifolds) and introduce on Minkowski and (pseudo) Riemannian spaces some types of Lagrange–Finsler parametrizations of metrics and connections because only using such variables we can construct general exact off–diagonal solutions in gravity and formulate an unified mathematical approach to classical field interactions and random processes with mixed holonomic/ nonholonomic / stochastic components.

1.1 Related directions

In order to avoid possible ambiguities with the formalism and terminology used in different approaches, we mention here some alternative methods with stochastic/ diffusion in curved spaces and random variables in gravity:

1. The theory of stochastic semiclassical gravity (in brief, *stochastic gravity*) is based on the Einstein–Langeven equation with additional sources due to the noise kernel. It was naturally constructed from semiclassical gravity and quantum field theory in curved spacetimes and non–equilibrium statistical mechanics, see a comprehensive review [26] and references therein. For short, stochastic gravity includes also its fluctuations, for instance, of quantum fields in curved spaces and explore how the metric fluctuations are induced and seed the structure of the universe, how such processes affect the back reaction of Hawking radiance in black hole dynamics and the black hole horizons, trans–Planckian physics etc.
2. The theory of stochastic (diffusion) equations on curved manifolds with local Euclidian signature[4] is a well developed topic (see e.g. [23, 24, 25]), see also related applications to locally anisotropic kinetic processes, gravity and astrophysics [11, 12] and an alternative approach to anisotropic diffusion [32]. For such constructions, random processes were considered on classical (manifolds) inducing possible matter field sources for gravitational filed equations.

Here we also note that various stochastic methods were also applied and developed for quantizing gauge and gravitational fields [33, 34, 35], see also a recent attempt for stochastic quantization of Hořava–Lifshitz gravity [36]. In this work, we shall restrict our considerations only to nonlinearly defined classical (non–quantum) gravitational processes generating exact solutions of the Einstein (not obligatory in the form of Einstein–Langeven) equations.

1.2 Goals of the paper:

Conventionally, there are three main purposes:

1. To provide an introduction into the theory of stochastic processes on nonolonomic Euclidian and Riemannian manifolds.
2. To generalize the theory of stochastic equations in relativistic form for nonholonomic (pseudo) Riemannian manifolds in a form allowing to study mixed sure (non–random) and stochastic gravitational processes.
3. To prove that prescribing stochastic generating and integration functions for "formal" general solutions of Einstein equations we define three types (so–called, horizontal, vertical and horizontal–vertical ones) of stochastic (nonholonomic) spacetimes. It will be shown how to construct general stochastic gravitational solutions with Killing symmetries and extensions to non–Killing ones and formulated the criteria when the non–stochastic solutions may transform into some stochastic ones, and inversely.

We organize the exposition as follows: In section 2 we summarize the Itô and Stratonovich stochastic calculus and diffusion theory on nonholonomic Euclidian and Riemannian manifolds. We use a synthesis of geometric and stochastic methods originally elaborated for the diffusion theory on nonholonomic vector bundles spaces and in Lagrange–Finsler geometry. Section 3 is devoted to the theory of nonholonomic special and general relativistic diffusion. A formalism of adapting the constructions to the nonlinear connection structure play a key role in distinguishing nonlinear gravitational and random gravitational processes and keeping certain similarity with former constructions for phase velocity spaces and moving frame method. In section 4, we generalize the anholonomic deformation method of constructing exact solutions in gravity in order to include into the scheme the possibility to generate stochastic metrics and nonlinear and linear connections. We also analyze explicit conditions/criteria when such stochastic gravitational fields are of vacuum type, with gravitational polarizations and/or cosmological constants and state the possibility to extend the constructions for nontrivial matter sources. A summary of results and concluding remarks are given in section 5.

2 Stochastic Processes and Nonholonomic Manifolds

In this section, we provide an introduction into the theory of stochastic differential equations on nonholonomic (pseudo) Riemannian manifolds [17]. We assume that the reader is familiar with the concepts and basic results on stochastic calculus, Brwonian motion and stochastic processes "rolled" on curved (usually, Riemannian) spaces [23, 24, 25].

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[4] in this paper we shall use equivalently the terms "space" and/or "spacetimes" for certain (pseudo/semi) Riemannian geometries, endowed with metrics of necessary signature, nonholonomic distributions with associated anholonomic frames, and (auxiliary) generalized connections

2.1 Geometric preliminaries

We consider a 4–d manifold (space, or spacetime) $\mathbf{V}$ endowed with a metric

$$\mathbf{g} = g_{\alpha\beta}(u^\gamma)du^\alpha \otimes du^\beta \tag{1}$$

of necessary signature $(\pm, \pm, \pm, \pm)$ when local coordinates are parametrized in the form $u^\alpha = (x^i, y^a)$, where $x^i = (x^1, x^2)$ and $y^a = (y^3 = v, y^4 = y)$.[5] Indices $i, j, k, \ldots = 1, 2$ and $a, b, c, \ldots = 3, 4$ are for a conventional $(2+2)$–splitting of dimension when the general (small Greek) abstract/coordinate indices when $\alpha, \beta, \ldots$ run values $1, 2, 3, 4$.

For our purposes, we split a metric (1) in the form

$$\mathbf{g} = g_{ij}dx^i \otimes dx^j + h_{ab}(dy^a + N_k^a dx^k) \otimes (dy^b + N_k^b dx^k), \tag{2}$$

and parametrize as

$$^\eta\mathbf{g} = \eta_i(x^k, v) \,^\circ g_i(x^k, v)dx^i \otimes dx^i + \eta_a(x^k, v) \,^\circ h_a(x^k, v)\mathbf{e}^a \otimes \mathbf{e}^a, \tag{3}$$
$$\mathbf{e}^3 = dv + \eta_i^3(x^k, v) \,^\circ w_i(x^k, v)dx^i, \quad \mathbf{e}^4 = dy^4 + \eta_i^4(x^k, v) \,^\circ n_i(x^k, v)dx^i.$$

In (3), we consider that $g_{ij} = diag[g_i = \eta_i \,^\circ g_i]$ and $h_{ab} = diag[h_a = \eta_a \,^\circ h_a]$ and $N_k^3 = w_i = \eta_i^3 \,^\circ w_i$ and $N_k^4 = n_i = \eta_i^4 \,^\circ n_i$. The gravitational 'polarizations' η_α and η_i^a determine nonholonomic deformations of metrics, $^\circ\mathbf{g} = [\,^\circ g_i, \,^\circ h_a, \,^\circ N_k^a] \rightarrow \,^\eta\mathbf{g} = [\,g_i, h_a, N_k^a]$ and can be defined by functions of necessary smooth class and/or any random (stochastic) variables.[6]

We use a boldface symbol $\mathbf{V}$ for nonholonomic manifolds (equivalently, spaces) when a Whitney splitting of tangent space $T\mathbf{V}$ is defined,

$$T\mathbf{V} = h\mathbf{V} \oplus v\mathbf{V}, \tag{4}$$

with conventional horizontal (h) and vertical (v) subspaces, respectively, $h\mathbf{V}$ and $v\mathbf{V}$. Such a geometric object defines a nonlinear connection (N–connection) structure $\mathbf{N}$, stated locally by a set of coefficients $\{N_k^a\}$ with respect to a corresponding coordinate basis. It should be emphasized here that we can introduce on (pseudo) Riemannian manifolds various types of nonholonomic distributions and N–connections, for instance, with 2+2 splitting, following the principle of covariance allowing us to consider any equivalently any system of reference/coordinates.

A N–connection structure $\mathbf{N} = \{N_k^a\}$ is present in (2) via N–adapted frame, $\mathbf{e}_\alpha$, and dual frame, $\mathbf{e}^\beta$, structures (i.e. N–elongated partial derivatives, respectively, differentials)

$$\mathbf{e}_\alpha \doteq \left(\mathbf{e}_i = \partial_i - N_i^a\partial_a, e_b = \partial_b = \frac{\partial}{\partial y^b} \right), \tag{5}$$

$$\mathbf{e}^\beta \doteq \left(e^i = dx^i, \mathbf{e}^a = dy^a + N_i^a dx^i \right). \tag{6}$$

Such local bases satisfy nonholonomic relations of type

$$[\mathbf{e}_\alpha, \mathbf{e}_\beta] = \mathbf{e}_\alpha \mathbf{e}_\beta - \mathbf{e}_\beta \mathbf{e}_\alpha = \mathbf{w}^\gamma{}_{\alpha\beta}(u)\,\mathbf{e}_\gamma,$$

with nontrivial anholonomy coefficients $\mathbf{w}^\alpha_{\beta\gamma}(u)$,

$$\mathbf{w}^a{}_{ji} = -\mathbf{w}^a{}_{ij} = \Omega_{ij}^a = \mathbf{e}_j N_i^a - \mathbf{e}_i N_j^a, \quad \mathbf{w}^b{}_{ia} = -\mathbf{w}^b{}_{ai} = \partial_a N_i^b$$

which is also related to the concept of nonholonomic manifold [17,18], i.e. a manifold endowed with a non–integrable distribution (in particular, with N–connection structure).

On a spacetime $\mathbf{V}$, there is an infinite number of metric compatible linear connections D, satisfying the conditions $D\mathbf{g} = 0$, and completely defined by a metric $\mathbf{g}$ (2). A subclass of such linear connections can be adapted to a chosen N–connection structure $\mathbf{N}$, when the splitting (4) is preserved under parallelism, and called distinguished connections (in brief, d–connections).[7] A general d–connection is denoted $\mathbf{D} = (hD, vD)$, being distinguished into, respectively,

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[5] In this paper, we shall consider metrics with local signature $(+, +, -, +)$ and/or $(+, +, +, -)$ in order to follow conventions from our previous results on exact solutions with local anisotropy [17,18].

[6] Such transforms, with deformations of the frame, metric, connections and other fundamental geometric structures, are more general than those considered for the Cartan's moving frame method when the geometric objects are re–defined equivalently with respect to various classes of systems of reference.

[7] On spaces endowed with N–connection structure, there are used the terms distinguished vectors/ forms / tensors etc (in brief, d–vectors, d–forms, d–tensors etc) in order to emphasize that the geometric constructions are adapted to the N–connection structure, i.e. preserving h– and v–splitting.

h- and v–covariant derivatives, hD and vD. We proved [17,18] that the Einstein equations in various gravity theories decouple and became integrable in very general forms for the so–called canonical d–connection, $\widehat{\mathbf{D}}$. With respect to N–adapted bases (5) and (6), $\widehat{\mathbf{D}}$ is computed to have the coefficients $\widehat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta} = (\widehat{L}^i_{jk}, \widehat{L}^a_{bk}, \widehat{C}^i_{jc}, \widehat{C}^a_{bc})$, for $h\widehat{D} = \{\widehat{L}^i_{jk}, \widehat{L}^a_{bk}\}$ and $v\widehat{D} = \{\widehat{C}^i_{jc}, \widehat{C}^a_{bc}\}$, with

$$\widehat{L}^i_{jk} = \frac{1}{2}g^{ir}\left(e_k g_{jr} + e_j g_{kr} - e_r g_{jk}\right), \ \widehat{L}^a_{bk} = e_b(N^a_k) + \frac{1}{2}h^{ac}\left(e_k h_{bc} - h_{dc}\,e_b N^d_k - h_{db}\,e_c N^d_k\right), \tag{7}$$

$$\widehat{C}^i_{jc} = \frac{1}{2}g^{ik}e_c g_{jk}, \ \widehat{C}^a_{bc} = \frac{1}{2}h^{ad}\left(e_c h_{bd} + e_c h_{cd} - e_d h_{bc}\right).$$

The d–connection $\widehat{\mathbf{D}}$ and its torsion $\widehat{\mathcal{T}} = \{\widehat{\mathbf{T}}^{\gamma}{}_{\alpha\beta} \equiv \widehat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta} - \widehat{\mathbf{\Gamma}}^{\gamma}{}_{\beta\alpha}; \widehat{T}^i_{jk}, \widehat{T}^i_{ja}, \widehat{T}^a_{ji}, \widehat{T}^a_{bi}, \widehat{T}^a_{bc}\}$, where the nontrivial coefficients

$$\widehat{T}^i_{jk} = \widehat{L}^i_{jk} - \widehat{L}^i_{kj}, \widehat{T}^i_{ja} = \widehat{C}^i_{jb}, \widehat{T}^a_{ji} = -\Omega^a_{ji}, \ \widehat{T}^c_{aj} = \widehat{L}^c_{aj} - e_a(N^c_j), \widehat{T}^a_{bc} = \widehat{C}^a_{bc} - \widehat{C}^a_{cb}, \tag{8}$$

are completely defined by the coefficients of metric $\mathbf{g}$ (2) following the conditions that $\widehat{\mathbf{D}}\mathbf{g} = 0$ and the "pure" horizontal and vertical torsion coefficients are zero, i. e. $\widehat{T}^i_{jk} = 0$ and $\widehat{T}^a_{bc} = 0$.

Any geometric construction for the canonical d–connection $\widehat{\mathbf{D}}$ can be re–defined equivalently into a similar one with the Levi–Civita connection following formula

$$\Gamma^{\gamma}{}_{\alpha\beta} = \widehat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta} + Z^{\gamma}{}_{\alpha\beta}, \tag{9}$$

where the distortion tensor $Z^{\gamma}{}_{\alpha\beta}$ is constructed in a unique form[8] from the coefficients of a metric $\mathbf{g}_{\alpha\beta}$.

2.2 $h-$ and $v-$adapted Euclidian diffusion

A nonholonomic manifold with conventional h- and v–splitting is with local fibered structure and similar to a vector/tangent bundle enabled with N–connection structure. In this work, we develop the approach in relativistic form (for holonomic spaces with local pseudo–Euclidean signature when the curved spacetime $\mathbf{V}$ has a tangent space with splitting of dimension $n + m$, for $n, m \geq 2$.

A distinguished Wiener process (in brief, Wiener d–process) of dimension $n + m$ is defined locally by a couple of elementary (Wiener) h– and v–processes $\mathcal{W}^{\alpha}(\tau) = \left(\mathcal{W}^i(\tau), \mathcal{W}^a(\tau)\right)$, where τ (in particular, we can take $\tau = t$ to be a time like parameter).[9] We consider a random (stochastic) curve on $\mathbf{V}$ lifted to the horizontal curve on the frame of orthonormalized bundles $O(\mathbf{V})$ related by frame transforms, $\mathbf{e}_{\alpha'} = \mathbf{e}^{\alpha}_{\alpha'}(u)\partial_{\underline{\alpha}} = \mathbf{e}^{\alpha}_{\alpha'}(u)\mathbf{e}_{\alpha}$, to N–adapted bases $\mathbf{e}_{\alpha}$ (5) when by $\partial_{\underline{\alpha}} = \partial/\partial u^{\underline{\alpha}} = (\partial_{\underline{i}} = \partial/\partial x^i, \partial_{\underline{a}} = \partial/\partial y^{\underline{a}})$ we denote a local coordinate base (if it will be necessary, we shall underline indices for coordinate bases; primed indices will be used for coordinates with respect to orthonormalized frames of reference).

2.2.1 Itô d–calculus

A diffusion distinguished process (d–process) in an Euclidian space is described by a couple of horizontal and vertical stochastic differential equations,

$$d\mathbf{U}^{\alpha} = \sigma^{\alpha}_{\alpha'}(\tau, \mathbf{U})\delta\mathcal{W}^{\alpha'} + b^{\alpha}(\tau, \mathbf{U})d\tau \tag{10}$$

where $\mathbf{U} = (hU, vU) \in \mathbb{R}^{n+m}$ is a stochastic d–process with $\mathbf{U}(0) = \mathbf{u}$, for $\mathbf{u} = \{u^{\beta} = (x^j, y^c)\}$, with parameter (time like variable, or temperature, $\tau \geq 0$). The given values $\sigma^{\alpha}_{\alpha'}$ and b^{α} are called respectively the diffusion coefficients

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[8] we can see this from explicit formulas

$$Z^a_{jk} = -\widehat{C}^i_{jb}g_{ik}h^{ab} - \frac{1}{2}\Omega^a_{jk}, \ Z^i_{bk} = \frac{1}{2}\Omega^c_{jk}h_{cb}g^{ji} - \Xi^{ih}_{jk}\widehat{C}^j_{hb}, \ Z^a_{bk} = \ ^+\Xi^{ab}_{cd}\widehat{T}^c_{kb}, \ Z^i_{kb} = \frac{1}{2}\Omega^a_{jk}h_{cb}g^{ji} + \Xi^{ih}_{jk}\widehat{C}^j_{hb}, \ Z^a_{jk} = 0,$$

$$Z^a_{jb} = -\ ^-\Xi^{ad}_{cb}\widehat{T}^c_{jd}, \ Z^a_{bc} = 0, \ Z^i_{ab} = -\frac{g^{ij}}{2}\left[\widehat{T}^c_{ja}h_{cb} + \widehat{T}^c_{jb}h_{ca}\right],$$

for $\Xi^{ih}_{jk} = \frac{1}{2}(\delta^i_j\delta^h_k - g_{jk}g^{ih})$ and $\ ^{\pm}\Xi^{ab}_{cd} = \frac{1}{2}(\delta^a_c\delta^b_d + h_{cd}h^{ab})$

[9] In Euclidian space, the coefficients of a $(n + m)$–dimensional such Wiener process $d\mathcal{W}^{\alpha} = \mathcal{W}^{\alpha}(\tau + \triangle\tau) - \mathcal{W}^{\alpha}(\tau)$ are defined for the probability density $\mathcal{P}(\mathcal{W}^{\alpha}) = \frac{1}{\sqrt{2\rho\pi\triangle\tau}}\exp\left(-\frac{[\mathcal{W}^{\alpha}(\tau)]^2}{2\rho\triangle\tau}\right)$ when the expectations $\langle\mathcal{W}^{\alpha}\rangle = 0$ and $\langle\mathcal{W}^{\alpha}(\ ^1\tau)\mathcal{W}^{\beta}(\ ^1\tau + \ ^2\tau)\rangle = \rho(\ ^2\tau)\,\delta^{\alpha\beta}$, for $\delta^{\alpha\beta}$ being the Kronecker symbol.

and the drift coefficients. It is possible to transform (10) into an integral equation $\mathbf{U}_\tau^\alpha = \mathbf{U}_0^\alpha + \int_0^\tau \sigma_{\alpha'}^\alpha(\varsigma, \mathbf{U})\delta\mathcal{W}_\varsigma^{\alpha'} + \int_0^\tau b^\alpha(\varsigma, \mathbf{U})d\varsigma$, i.e. the Itô stochastic integral adapted to h- and v–splitting, defining an Itô process as a Markovian process (see details in [23, 24, 25, 8]). In the above formulas we write $\delta\mathcal{W}_\varsigma^{\alpha'}$ instead of $d\mathcal{W}_\varsigma^{\alpha'}$ in order to emphasize that the approach is with N–elongated partial derivatives and differentials, (5) and (6), instead of usual ones.

If $\mathbf{U}_\tau^\alpha$ is an Itô process, then a function $\mathbf{Y}_\tau^\alpha = f(\mathbf{U}_\tau^\alpha)$ is also an Itô process when the Itô N–adapted formula for stochastic differential $\delta f = \mathbf{A}f$ with associated diffusion d–operator $\mathbf{A} = hA \oplus vA$,

$$\mathbf{A} = \frac{\rho}{2}\sum_{\alpha'=1}^{n+m}\{\frac{1}{2}\sigma_{\alpha'}^i(\tau, \mathbf{U})\sigma_{\alpha'}^j(\tau, \mathbf{U})\left(\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i\right)f + \sigma_{\alpha'}^a(\tau, \mathbf{U})\sigma_{\alpha'}^b(\tau, \mathbf{U})e_ae_bf + b^\alpha(\tau, \mathbf{U})\mathbf{e}_\alpha f\}, \tag{11}$$

where, for instance, $hA = \frac{\rho}{2}\sum_{\alpha'=1}^n\{\frac{1}{2}\sigma_{\alpha'}^i(\tau, \mathbf{U})\sigma_{\alpha'}^j(\tau, \mathbf{U})(\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i)f + b^i(\tau, \mathbf{U})\mathbf{e}_if\}$. Such an operator contains the second derivative and the Itô stochastic d–differential δf does not satisfy usual properties for linear operators which makes the theory more sophisticate.

For any stochastic N–adapted process $\mathbf{U}_\tau^\alpha \in \mathbb{R}^{n+m}$, we can introduce the probability density function $\phi(\tau, \mathbf{u})$ and determine how it evolves with time/temperature parameter τ. This function allows us to compute the probability Pr that a realization of a set of variables $U_1, ... U_{n+m}$ falls inside a domain $\mathcal{U}$ in the $(n+m)$–dimensional space of such variables,

$$\Pr(\mathbf{U}_\tau^\alpha \in \mathcal{U}) := \int_\mathcal{U} \phi(\tau, \mathbf{u})dx^1...dx^n\delta y^{n+1}...\delta y^{n+m}. \tag{12}$$

We can calculate the expected value ${}^uE[f(\mathbf{U}_\tau)]$ for any function[10] f of $\mathbf{U}$,

$$ {}^uE[f(\mathbf{U}_\tau)] := \int_\mathcal{U} f(\mathbf{u})\phi(\tau, \mathbf{u})\delta u^1...\delta u^{n+m}. \tag{13}$$

If we define $\underline{f}(\tau, \mathbf{u}) := {}^uE[f(\mathbf{U}_\tau)]$, such a function is subjected to the condition (in literature, it is called the Focker–Plank, or the forward Kolmogorov, equation)

$$\partial_\tau \underline{f}(\tau, \mathbf{u}) = \mathbf{A}\underline{f}(\tau, \mathbf{u}), \tag{14}$$
$$\underline{f}(0, \mathbf{u}) = f(\tau, \mathbf{u}),$$

where $\partial_\tau = \partial/\partial\tau$ and the d–operator $\mathbf{A}$ is defined by (11). Physical applications are usually considered following exact/approximate solutions of such equations.[11]

Finally, we provide the formula $\mathbf{A}f = \lim_{\tau\to 0}\frac{{}^uE[f(\mathbf{U}_\tau)]-f(\mathbf{u})}{\tau}$, defined for any suitable function f, where $\mathbf{u} = \mathbf{U}_{\tau=0}$ is the initial point of N–adapted stochastic process $\mathbf{U}_\tau$. Under the conditions that above integral formulas hold true, we can say that a diffusion generator/d–operator $\mathbf{A}$, and its adjoint ${}^*\mathbf{A}$, of $\mathbf{U}_\tau$ is associated to an Itô d–process.

2.2.2 Stratonovich d–calculus

There is an equivalent reformulation of the stochastic calculus by using the Stratonovich integral[12], which is more convenient for various curved spacetime generalizations. In N–adapted form, we write

$$d\mathbf{U}^\alpha = \widetilde{\sigma}_{\alpha'}^\alpha(\tau, \mathbf{U}) \circ \delta\mathcal{W}^{\alpha'} + \widetilde{b}^\alpha(\tau, \mathbf{U})d\tau,$$

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[10] in physical applications, we can consider a smooth class or any class for which a corresponding integration procedure is defined

[11] Similar constructions are possible for the (Hermitian adjoint) d–operator ${}^*\mathbf{A}$ of $\mathbf{A}$ (the formulas can be proven using (12) and (13)), $\partial_\tau\phi(\tau, \mathbf{u}) = {}^*\mathbf{A}\phi(\tau, \mathbf{u})$; for arbitrary function $f(\mathbf{u})$, $\int_\mathcal{U} f(\mathbf{u}) {}^*\mathbf{A}\phi(\tau, \mathbf{u})du^1...du^{n+m} = \int_\mathcal{U} f(\mathbf{u})\partial_\tau\phi(\tau, \mathbf{u})du^1...du^{n+m}$, for ${}^*\mathbf{A} = \frac{\rho}{2}\sum_{\alpha'=1}^{n+m}\{\frac{1}{2}\sigma_{\alpha'}^i(\tau, \mathbf{U})\sigma_{\alpha'}^j(\tau, \mathbf{U})\left(\mathbf{e}_i\mathbf{e}_j + \mathbf{e}_j\mathbf{e}_i\right)f + \sigma_{\alpha'}^a(\tau, \mathbf{U})\sigma_{\alpha'}^b(\tau, \mathbf{U})e_ae_bf - \mathbf{e}_\alpha b^\alpha(\tau, \mathbf{U})f\}$.

[12] even such integrals do not result in Markovian processes

where we put $"\sim"$ and $"\circ"$ in order to not confuse this interpretation with that given by (10) for the Itô approach. To get equivalent formulations of stochastic calculus is possible if we identify

$$\widetilde{\sigma}_{\alpha'}^{\alpha}(\tau, \mathbf{U}) = \sigma_{\alpha'}^{\alpha}(\tau, \mathbf{U}), \ \widetilde{b}^{\alpha}(\tau, \mathbf{U}) = b^{\alpha}(\tau, \mathbf{U}) - \frac{\rho}{2} \sum_{\alpha'=1}^{n+m} \sigma_{\alpha'}^{\beta}(\tau, \mathbf{U}) \mathbf{e}_{\beta} \sigma_{\alpha'}^{\alpha}(\tau, \mathbf{U}).$$

Such a re–definition of drift coefficients results in a linear (on partial derivatives) operator for the associated diffusion/generator d–operator (compare, for instance, with (11) where there are contained second partial derivatives), for the Stratonovich interpretation written in the form

$$\widetilde{\mathbf{A}} = \frac{\rho}{2} \sum_{\alpha'=1}^{n+m} \mathbf{L}_{\alpha'} \mathbf{L}_{\alpha'} + \mathbf{L}_0, \tag{15}$$

where $\mathbf{L}_{\alpha'} = \sigma_{\alpha'}^{\beta}(\tau, \mathbf{u}) \mathbf{e}_{\beta}$ and $\mathbf{L}_0 = \widetilde{b}^{\alpha}(\tau, \mathbf{u}) \mathbf{e}_{\beta}$ are called the fundamental (for a diffusion d–process) d–vector fields.

The associated N–adapted Focker–Plank equation (compare with (14)) in the Stratonovich approach is

$$\partial_{\tau} \phi(\tau, \mathbf{u}) = \frac{\rho}{2} \sum_{\alpha'=1}^{n+m} \mathbf{e}_{\beta} \sigma_{\alpha'}^{\beta}(\tau, \mathbf{u}) \mathbf{e}_{\gamma} [\sigma_{\alpha'}^{\gamma}(\tau, \mathbf{u}) \phi(\tau, \mathbf{u})] - \mathbf{e}_{\alpha} \widetilde{b}^{\alpha}(\tau, \mathbf{u}) \phi(\tau, \mathbf{u}).$$

2.3 Diffusion on nonholonomic manifolds

The theory of stochastic differential equations on a nonholonomic manifold $\mathbf{V}$ enabled with a metric compatible d–connection can be constructed similarly to the case of $n + m$ dimensional Riemannian spaces if we work with N–adapted frames and co–frames (5) and (6). We use N–adapted variables/coordinates $\mathbf{r} = (\mathbf{u}, \mathbf{e}) = (u^{\alpha}, \mathbf{e}_{\beta'}^{\beta'}) \in O(\mathbf{V})$ and their frame/coordinate transforms. The infinitesimal motion of a smooth curve $u^{\alpha}(\tau) \in \mathbf{V}$ is lifted naturally to that of $\gamma^{\alpha'}(\tau) \in O(\mathbf{V})$ using the ordinary differential equations for N–adapted to (4) parallel transport,

$$\delta u^{\alpha} = \mathbf{e}_{\alpha'}^{\alpha}(u^{\beta}) \delta \gamma^{\alpha'} \text{ and } \delta \mathbf{e}_{\alpha'}^{\alpha}(u^{\mu}) = -\widehat{\mathbf{\Gamma}}_{\beta\nu}^{\alpha}(u^{\mu}) \mathbf{e}_{\alpha'}^{\nu}(u^{\mu}) \delta u^{\beta},$$

where we use symbols $\delta u^{\alpha}, \delta \mathbf{e}_{\alpha'}^{\alpha}$ etc instead of respective $du^{\alpha}, d\mathbf{e}_{\alpha'}^{\alpha}...$, in order to emphasize that in our constructions we use N–elongated partial derivatives and differentials. The coefficients of the canonical d–connection $\widehat{\mathbf{\Gamma}}_{\beta\nu}^{\alpha}$ are computed following formulas (7).

A curve $\mathbf{r}(\tau) = (\mathbf{u}(\tau), \mathbf{e}(\tau))$ is considered as the horizontal lift (it should be not confused with the h–splitting considered in previous sections) of a curve $\mathbf{u}(\tau)$ to the nonholonomic bundle $O(\mathbf{V})$ (modelled by atlases of carts covered with Euclidean spaces $\mathbb{R}^{(n+m)^2+n+m}$ and a "horizontal" curve $\gamma^{\alpha'}(\tau)$ in the tangent spaces, which can be identified locally with an Euclidean spaces $\mathbb{R}^{n+m}$). Using such lifts, we can define stochastic differential equations on nonholonomic manifolds when the fundamental Wiener processes are associated to h– and v–components and corresponding Euclidean carts. The corresponding stochastic integrals are defined in the sense of Stratonovich on any such open regions of $\mathbf{V}$ and $O(\mathbf{V})$ when the canonical realization of multidimensional Wiener processes is used and $\delta \gamma^{\alpha'}(\tau) \to \delta \mathcal{W}^{\alpha'}$. The stochastic differential equation describing N–adapted diffusion on a nonholonomic manifold is

$$\delta u^{\alpha} = \mathbf{e}_{\alpha'}^{\alpha}(\tau) \circ \delta \mathcal{W}^{\alpha'} + \mathbf{A}^{\alpha}(\tau) d\tau \tag{16}$$
$$\delta \mathbf{e}_{\alpha'}^{\alpha}(\tau) = -\widehat{\mathbf{\Gamma}}_{\beta\nu}^{\alpha}(\tau) \mathbf{e}_{\alpha'}^{\nu}(\tau) \circ \delta u^{\beta},$$

where the components of d–vector $\mathbf{A}^{\alpha}(\tau) = (A^i(\tau), A^a(\tau))$ are introduced additionally in order to model various types of stochastic processes and take into account possible external forces. Here $\delta^{\alpha'\beta'} \mathbf{e}_{\alpha'}^{\alpha}(\mathbf{u}(\tau)) \mathbf{e}_{\beta'}^{\beta}(\mathbf{u}(\tau)) = \mathbf{g}^{\alpha\beta}$ with $\delta^{\alpha'\beta'}$ taken as the flat Euclidean metric splitting as $\delta^{\alpha'\beta'} = \left(\delta^{i'j'}, \delta^{a'b'} \right)$.

Extending on $O(\mathbf{V})$ the definition of fundamental d–vector fields from (15), $\mathbf{L}_{\alpha'} \to {}^{O}\mathbf{L}_{\alpha'}$ and $\mathbf{L}_0 \to {}^{O}\mathbf{L}_0$, where

$${}^{O}\mathbf{L}_{\alpha'} = \mathbf{e}_{\alpha'}^{\alpha} \mathbf{e}_{\alpha} - \widehat{\mathbf{\Gamma}}_{\beta\nu}^{\alpha}(u^{\mu}) \mathbf{e}_{\alpha'}^{\beta} \mathbf{e}_{\beta'}^{\nu} \frac{\partial}{\mathbf{e}_{\beta'}^{\beta}}, \ {}^{O}\mathbf{L}_0 = \mathbf{A}^{\alpha}(\tau, \mathbf{U}) \mathbf{e}_{\alpha} - \widehat{\mathbf{\Gamma}}_{\beta\nu}^{\alpha} \mathbf{A}^{\beta} \mathbf{e}_{\beta'}^{\nu}(\tau) \frac{\partial}{\mathbf{e}_{\beta'}^{\beta}},$$

we provide a nonholonomic generalization (horizontal lift of the diffusion d–operator ${}^{\mathbf{V}}\widetilde{\mathbf{A}}$) of the diffusion generator $\widetilde{\mathbf{A}}$ on the bundle of orthonormalized N–adapted frames,

$${}^{O}\widetilde{\mathbf{A}} = \frac{\rho}{2} \sum_{\alpha'=1}^{n+m} {}^{O}\mathbf{L}_{\alpha'} \ {}^{O}\mathbf{L}_{\alpha'} + \ {}^{O}\mathbf{L}_0. \tag{17}$$

For any projection of a function f in $O(\mathbf{V})$ to $\mathbf{V}$ (when, for instance, $f(\mathbf{r}) = f(\mathbf{u}, 0)$, $\mathbf{r} = (u^\alpha, \mathbf{e}^{\beta'}_{\beta})$), we can write

$$^O\widetilde{\mathbf{A}}f(\mathbf{r}) = {}^{\mathbf{V}}\widetilde{\mathbf{A}}f(\mathbf{u}),$$

$$\text{where} \quad {}^{\mathbf{V}}\widetilde{\mathbf{A}} = \frac{\rho}{2}\sum_{\alpha'}\mathbf{e}^\alpha_{\alpha'}\mathbf{e}_{\alpha}(\mathbf{e}^\beta_{\alpha'}\mathbf{e}_{\beta}) + \mathbf{A}^\beta\,\mathbf{e}_{\beta} = \frac{\rho}{2}\widehat{\triangle} + \mathbf{A}^\beta\,\mathbf{e}_{\beta} \tag{18}$$

$$\text{and} \quad \widehat{\triangle} = \frac{1}{2}\mathbf{g}^{\alpha\beta}\left[\mathbf{e}_{\alpha}\mathbf{e}_{\beta} + \mathbf{e}_{\beta}\mathbf{e}_{\alpha} + \left(\widehat{\mathbf{\Gamma}}^\nu_{\alpha\beta} + \widehat{\mathbf{\Gamma}}^\nu_{\beta\alpha}\right)\mathbf{e}_{\nu}\right] \tag{19}$$

is the canonical Laplace–Beltrami d–operator defined by the inverse coefficients $\mathbf{g}^{\alpha\beta}$ of d–metric (2), the canonical d–connection $\widehat{\mathbf{\Gamma}}^\nu_{\alpha\beta}$ (7) and N–connection $\mathbf{N} = \{N^a_k\}$ (4).

The operators ${}^{\mathbf{V}}\widetilde{\mathbf{A}}$ (18) and $\widehat{\triangle}$ (19) allows us to formulate a generalized Kolmogorov backward equation on a nonholonomic manifold $\mathbf{V}$,

$$\partial_\tau \underline{f}(\tau, \mathbf{u}) = {}^{\mathbf{V}}\widetilde{\mathbf{A}}\underline{f}(\tau, \mathbf{u}), \tag{20}$$
$$\underline{f}(0, \mathbf{u}) = f(\mathbf{u}).$$

Since the canonical Laplace–Beltrami d–operator is self–adjoint, $\widehat{\triangle} = {}^*\widehat{\triangle}$, we can construct a self–adjoint ${}^*\left({}^{\mathbf{V}}\widetilde{\mathbf{A}}\right)$ to ${}^{\mathbf{V}}\widetilde{\mathbf{A}}$ as we explained above in footnote 11. As a result, the corresponding generalized Fokker–Planck equation on $\mathbf{V}$ is

$$\partial_\tau F = -\frac{1}{\sqrt{|\mathbf{g}_{\alpha\beta}|}}\mathbf{e}_{\nu}(\sqrt{|\mathbf{g}_{\alpha\beta}|}\mathbf{A}^\nu F) + \frac{\rho}{2}\widehat{\triangle}F, \tag{21}$$

where $F = F(\tau, {}^1\mathbf{u}; 0, {}^2\mathbf{u})$ is the transition probability with the initial condition $F(0, {}^1\mathbf{u}; 0, {}^2\mathbf{u}) = \delta({}^1\mathbf{u} - {}^2\mathbf{u})$ for any two points ${}^1\mathbf{u}, {}^2\mathbf{u} \in \mathbf{V}$ and adequate boundary conditions at infinity.[13] Finally we note that we get usual evolution/diffusion equations in nonholonomic curved spaces, $\partial_\tau F = \frac{\rho}{2}\widehat{\triangle}F$, if we impose the condition that the divergence in (21) (i.e. the first term in the right part of equation) is taken zero.

3 Nonholonomic Diffusion in General Relativity

In the derivation of relativistic diffusion equations and constructing gravity theories, we have to take into account in an appropriate way the fact that the speed of light has a constant maximal value. Geometrically, such a fundamental experimental fact is encoded into the special theory of relativity as the condition that the velocity space is a hyperboloid (i.e. a special type three dimensional, 3–d, Riemannian manifold) embedded into the 4–d velocity Minkowski space. A formal geometric analogy between Euclidian/Riemannian/Finsler etc geometries and respective ones with "pseudo" signatures can be preserved by introducing a formal "imaginary" time like in the "early" works on general relativity [45, 46].

In this paper, we elaborate a geometric and stochastic formalism in order to include in the scheme the relativistic diffusion processes with exact solutions for gravitational field equations. Such solutions can be constructed in general form only by imposing corresponding nonholonomic constraints on the systems of partial differential equations for dynamical and/or stochastic systems. The goal of this section is to consider an extension of the theory of relativistic diffusion on (flat) Minkowski and (curved) Einstein spaces when such spacetimes are enabled with conventional h– and v–splitting (respectively, with trivial and/or nontrivial N–connection structure). The values of coefficients metrics and connections are considered to be given from certain (necessary smooth class) solutions of classical gravitational and matter fields equations. In the next section 4, the scheme will be completed by elaborating a method of generating both classical and stochastic solutions of Einstein equations.

3.1 The special relativistic nonholonomic diffusion

The velocity space in special relativity is characterized by a noncompact hyperbolic structure which for the 4–d Minkowski spacetime ${}^3_1M$ is parametrized by a corresponding relation

$$- (v^1)^2 + (v^2)^2 + (v^3)^2 + (v^4)^2 = -1, \tag{22}$$

---

[13] The same equations can be considered for the probability density $\varphi(\tau, \mathbf{u})$ but for the initial condition $\varphi(\tau = 0, \mathbf{u}) = {}^0\varphi(\mathbf{u})$.

where v^α are normalized velocity variables defined for the typical fiber of tangent bundle $T(\,{}^3_1 M)$. We can introduce the laboratory time $t = \tau u^1/c$, where c is the light velocity and τ is the proper time, and express $v^1 = \left[1 + (v^2)^2 + (v^3)^2 + (v^4)^2\right]^{1/2}$. The local coordinates on $T(\,{}^3_1 M)$ can be parametrized in the form $u^\alpha = (x^i, y^a = v^a)$, where $i, j, ... = 1, 2, 3, 4$ and $a, b, c... = 5, 6, 7, 8$. It is considered a conventional $1+3$ splitting for $\,{}^3_1 M$. We shall write $\,{}^h T(\,{}^3_1 M)$ if the v–coordinates are subjected to constraints of type (22) and say that its typical fiber space is a hyperbolic velocity space (when the hyperboloid is embedded into the 4–d Minkovski spacetime). Such a space is naturally enabled with a corresponding metric, $h_{\widehat{a}\widehat{b}}(v^c)$, and linear connection (Christoffel connection coefficients on the hyperboloid), $\gamma^{\widehat{a}}_{\widehat{b}\widehat{c}}(v^e)$, when

$$h_{\widehat{a}\widehat{b}}(v^c) = \delta_{\widehat{a}\widehat{b}} - v^{\widehat{a}} v^{\widehat{b}}/(v^1)^2, \; \gamma^{\widehat{a}}_{\widehat{b}\widehat{c}}(v^e) = v^{\widehat{a}} h_{\widehat{b}\widehat{c}}, \tag{23}$$

where (in this subsection) $\widehat{a}, \widehat{b}, ... = 2, 3, 4$. We can prescribe additionally any nonholonomic $2+2$ and/or $4+4$ splitting on $\,{}^3_1 M$ and/or $T(\,{}^3_1 M)$ by prescribing on such spaces corresponding nonholonomic distributions with associated N–connection structures of type (4). A nonholonomic distribution in $T(\,{}^3_1 M)$ is given by $\mathbf{N} = \{N^{\widehat{a}}_k(u^{\widehat{\alpha}})\}$, local coordinates $\widehat{\mathbf{u}} = \{u^{\widehat{\alpha}} = (x^i, v^{\widehat{a}})\}$. The d–metric structure $\widehat{\mathbf{g}}$ on $T(\,{}^3_1 M))$ can be written (in a form similar to (2))

$$\widehat{\mathbf{g}} = \eta_{ij} dx^i \otimes dx^j + h_{\widehat{a}\widehat{b}}(dv^{\widehat{a}} + N^{\widehat{a}}_k dx^k) \otimes (dv^{\widehat{b}} + N^{\widehat{b}}_k dx^k), \tag{24}$$

where $\eta_{ij} = (-, +, +, +)$, for which, using formulas (7), we can compute the corresponding coefficients of the canonical d–connection $\widehat{\mathbf{\Gamma}}^{\widehat{\gamma}}_{\widehat{\alpha}\widehat{\beta}} = (\widehat{L}^i_{jk}, \widehat{L}^{\widehat{a}}_{\widehat{b}k}, \widehat{C}^i_{j\widehat{c}}, \widehat{C}^{\widehat{a}}_{\widehat{b}\widehat{c}})$.

For nonholonomic distributions with $N^{\widehat{a}}_k(u^{\widehat{\alpha}})$ when $\Omega^{\widehat{a}}_{ji} = 0$, see formulas (8), we get $\widehat{C}^{\widehat{a}}_{\widehat{b}\widehat{c}} = \gamma^{\widehat{a}}_{\widehat{b}\widehat{c}}(v^e)$. But, in general, we work with arbitrary frame and coordinate transforms and the fundamental geometric objects on $\,{}^h T(\,{}^3_1 M)$ with nontrivial h- and v–splitting induced by $\mathbf{N}$, when $\widehat{\mathbf{g}} = \{\,\widehat{\mathbf{g}}_{\widehat{\alpha}\widehat{\beta}} = [\widehat{g}_{ij}(u^{\widehat{\alpha}}), h_{\widehat{a}\widehat{b}}(u^{\widehat{\alpha}})]\}$. It is possible to introduce N–adapted orthonormalized frame bases $\mathbf{E}_{\widehat{\alpha}'} = \mathbf{E}^{\widehat{\alpha}}_{\widehat{\alpha}'}(\widehat{\mathbf{u}})\partial_{\underline{\widehat{\alpha}}} = \mathbf{E}^{\widehat{\alpha}}_{\widehat{\alpha}'}(\widehat{\mathbf{u}})\mathbf{E}_{\widehat{\alpha}}$ for which

$$\widehat{\mathbf{g}}_{\widehat{\alpha}'\widehat{\beta}'} = [\widehat{g}_{i'j'} = \eta_{i'j'}, h_{\widehat{a}'\widehat{b}'} = \delta_{\widehat{a}'\widehat{b}'}] = \mathbf{E}^{\widehat{\alpha}}_{\widehat{\alpha}'} \mathbf{E}^{\widehat{\beta}}_{\widehat{\beta}'} \, \widehat{\mathbf{g}}_{\widehat{\alpha}\widehat{\beta}}, \tag{25}$$

$$\widehat{g}_{i'j'} = E^i_{i'} E^j_{j'} \, \widehat{g}_{ij}, \; h_{\widehat{a}'\widehat{b}'} = E^{\widehat{a}}_{\widehat{a}'} E^{\widehat{b}}_{\widehat{b}'} \, h_{\widehat{a}\widehat{b}},$$

when the values with "un–primed" indices are given by the coefficients of d–metric (24) and (23). In our approach, the frame transform are written with capital letters, $\mathbf{E}^{\widehat{\alpha}}_{\widehat{\alpha}'} = [E^i_{i'}, E^{\widehat{a}}_{\widehat{a}'}]$, for spaces with pseudo–Euclidian and/or hyperbolic fiber/velocity space. The values $E^i_{i'}$ parametrize moving frames adapted to the position spacetime $\,{}^3_1 M$ and the values $E^{\widehat{a}}_{\widehat{a}'}(\tau)$ state moving velocity frames adapted to the typical fiber of $\,{}^h T(\,{}^3_1 M)$.

Working with "hyperbolic geometric data" $\widehat{\mathbf{g}}$ (24), $\widehat{\mathbf{\Gamma}}^{\widehat{\gamma}}_{\widehat{\alpha}\widehat{\beta}}$ and $\mathbf{E}^{\widehat{\alpha}}_{\widehat{\alpha}'}$ (25) for a prescribed N–connection structure $\mathbf{N} : TT(\,{}^3_1 M) = hT(\,{}^3_1 M) \oplus vT(\,{}^3_1 M)$, we can define a model of nonholonomic relativistic diffusion in special relativity (i.e. on total space $T(\,{}^3_1 M)$) similarly to the constructions provided in section 2.3. Let us consider the frame bundle space $\mathbf{F}(\,{}^h T(\,{}^3_1 M))$ with local coordinates $\widehat{\mathbf{r}} = \{u^{\widehat{\alpha}}, \mathbf{E}^{\widehat{\alpha}}_{\widehat{\alpha}'}\}$ and identify $dx^i(\tau) = v^i(\tau)d\tau$, where τ can be interpreted as an evolution parameter along the world lines of the particles which can be chosen as the proper time (we can take τ as the temperature in gravitational thermo–field models etc). The N–adapted relativistic stochastic equations on $\,{}^h T(\,{}^3_1 M)$ are similar to (16),

$$\delta u^{\widehat{\alpha}} = \mathbf{E}^{\widehat{\alpha}}_{\widehat{\alpha}'}(\tau) \circ \delta \mathcal{W}^{\widehat{\alpha}'} + \mathbf{A}^{\widehat{\alpha}}(\tau)d\tau \tag{26}$$

$$\delta \mathbf{E}^{\widehat{\alpha}}_{\widehat{\alpha}'}(\tau) = -\widehat{\mathbf{\Gamma}}^{\widehat{\alpha}}_{\widehat{\beta}\widehat{\nu}}(\tau)\mathbf{E}^{\widehat{\nu}}_{\widehat{\alpha}'}(\tau) \circ \delta u^{\widehat{\beta}},$$

when the velocity coordinates are subjected to the hyperbolicity conditions (22) and the components of drift d–vector are parametrized $\mathbf{A}^{\widehat{\alpha}}(\tau) = (A^i(\tau) = b^i(\tau), A^{\widehat{a}}(\tau) = B^{\widehat{a}}(\tau))$.

The stochastic differential equations (26) adapted to the nonholonomic hyperbolic velocity structure split into two families (respectively for the position coordinates and for the velocity type coordinates),

$$\delta x^i(\tau) = E^i_{i'}(\tau) \circ \delta \mathcal{W}^{i'} + b^i(\tau)d\tau \tag{27}$$

$$dE^i_{i'}(\tau) = -\widehat{L}^i_{jk}(\tau)E^k_{i'}(\tau) \circ dx^k,$$

and

$$\delta v^{\widehat{a}} = E^{\widehat{a}}_{\widehat{a}'}(\tau) \circ \delta \mathcal{W}^{\widehat{a}'} + B^{\widehat{a}}(\tau)d\tau \tag{28}$$

$$\delta E^{\widehat{a}}_{\widehat{a}'}(\tau) = -\widehat{C}^{\widehat{a}}_{\widehat{b}\widehat{c}}(\tau)E^{\widehat{c}}_{\widehat{a}'}(\tau) \circ \delta v^{\widehat{b}}.$$

where a nontrivial $B^{\widehat{a}}(\tau) = F^{\widehat{a}}/m_0$ is defined by spacial components of a 4–force F^a acting on particles of rest mass m_0. For trivial N–connection structure, the equations (27) and (28) transform respectively into relativistic stochastic equations (2) and (3) proposed in Ref. [8]. In our approach, we can introduce a nonholonomic dynamics in the velocity space of special relativity, via corresponding N–connection structure and generalized d–connections. This way we can model various theories with restricted/brocken Lorentz symmetry etc.

For simplicity, we omit here the considerations when from (26) a relativistic theory of $(\mathbf{A}, \mathbf{L})$–diffusion is derived. [14]

3.2 Nonholonomic diffusion and gravitational interactions

There are two classes of relativistic diffusion theories for gravity:

1. The first one is for modelling relativistic stochastic processes on fixed classical curved spacetimes. We have to consider, instead of the flat Minkovski spacetime ${}_{1}^{3}M$ and metric $\widehat{\mathbf{g}}$ (24), a (pseudo) Riemannian spacetime $\mathbf{V}$ and a solution $\mathbf{g}_{\mu\nu}$ in general relativity (GR). The four–velocity v^μ of massive particles in GR satisfies the condition

$$\mathbf{g}_{\mu\nu}(u^\alpha)v^\mu v^\nu = -1, \tag{29}$$

   (here $\mu, \nu = 1, 2, 3, 4$), which is a generalization of (22) (we suppose that in any point $\mathbf{u} \in \mathbf{V}$ the hyperbolicity condition holds in the typical fiber of $T\mathbf{V}$). By using the orthonormalized moving frames of the pseudo–Riemannian manifold we get the same formulas as for relativistic diffusion in special relativity, see details in Section 3 of [8]. If the Levi–Civita connection ∇ is changed into the canonical d–connection $\widehat{\mathbf{D}}$ (or, for other models, $\mathbf{D} = {}^{c}\mathbf{D}$, for instance, the Cartan d–connection), we get relativistic models on nonholonomic (pseudo) Riemann and/or Lagrange–Finsler spaces [11, 12].
2. The second class of theories is that when d–metrics $\mathbf{g} = \{\mathbf{g}_{\mu\nu}\}$ (2) are exact solutions of Einstein equations in a gravity theory for $\widehat{\mathbf{D}}$, or its restriction to ∇ (with $Z^\gamma_{\alpha\beta} = 0$ in formulas (9)), when some coefficients $\mathbf{g}_{\mu\nu}$ take stochastic values, which results in a relativistic gravitational diffusion of metrics in general relativity (for various generalizations). We work with nonholonomic gravitational configurations with associated N–connection splitting because in such cases we are able to solve the Einstein equations in general forms and analyze mutual diffusions of metrics (the splitting of equations is not possible for not N–adapted constructions with the Levi–Civita connection, see section 4).

In this section we shall elaborate a model of relativistic diffusion for a nonholonomic (pseudo) Riemannian spaces $\mathbf{V}$ with prescribed N–connection structure $N: \ T\mathbf{V} = h\mathbf{V} \oplus v\mathbf{V}$, when $\dim \mathbf{V} = 4$. The constructions are performed for the canonical d–connection $\widehat{\mathbf{D}}$ and d–metrics being solution of the nonholonomic Einstein equations with limits of type $\widehat{\mathbf{D}}_{|Z \to 0} \to \nabla$.

3.2.1 N–adapted frames and stochastic equations in GR

Orthonormalized N–adapted frames on $\mathbf{V}$, $\mathbf{e}_{\alpha'} = \mathbf{e}^{\underline{\alpha}}_{\alpha'}(\mathbf{u})\partial_{\underline{\alpha}} = \mathbf{e}^\alpha_{\alpha'}(\mathbf{u})\mathbf{e}_\alpha$, with $\mathbf{e}_\alpha$ being of type (5), can be defined by any nondegernerated matrix fields $\mathbf{e}^{\underline{\alpha}}_{\alpha'}(\mathbf{u})$ and $\mathbf{e}^\alpha_{\alpha'}(\mathbf{u})$ (and/or, respectively, their inverses, $\mathbf{e}_{\underline{\alpha}}^{\alpha'}(\mathbf{u})$ and $\mathbf{e}_\alpha^{\alpha'}(\mathbf{u})$) subjected to the conditions

$$\eta_{\alpha'\beta'} = \mathbf{g}_{\alpha\beta}\mathbf{e}^\alpha_{\alpha'}(\mathbf{u})\ \mathbf{e}^\beta_{\beta'}(\mathbf{u}), \tag{30}$$

where $\eta_{\alpha'\beta'} = (1, 1, -1, 1)$, and the space–time orientation of coordinates is chosen in a form which will be convenient for constructing exact generic off–diagonal solutions of Einstein equations (see next section).

We can compute the orthonormalized components $v^{\alpha'}$ of a 4–vector v^α, $v^{\alpha'} = \mathbf{e}_\alpha^{\alpha'}(\mathbf{u})v^\alpha$, when $\eta_{\alpha'\beta'}v^{\alpha'}v^{\beta'} = -1$ which can be proven using algebraic relations (29) and (30). By definition, $\widehat{\mathbf{D}}$ is metric compatible and we can impose the condition $\widehat{\mathbf{D}}_\beta \mathbf{e}^\alpha_{\alpha'} = 0$ and $\widehat{\mathbf{D}}_\beta \mathbf{e}_\alpha^{\alpha'} = 0$. We get such a transformation law for the canonical d–connection coefficients,

$$\widehat{\mathbf{\Gamma}}^{\alpha'}_{\beta\nu'}(\mathbf{u}) = \mathbf{e}_\alpha^{\alpha'}(\mathbf{u})\left(\widehat{\mathbf{\Gamma}}^\alpha_{\beta\nu}\mathbf{e}^\nu_{\nu'}(\mathbf{u}) + \mathbf{e}_\beta \mathbf{e}^\alpha_{\nu'}(\mathbf{u})\right),$$

---

[14] Corresponding formulas are similar to (18)–(21) with that difference that the "hyperbolic" small Greek and velocity indices, for this section, are with "hats" emphasizing the fact that the nonholonomic diffusion evolution is adapted to the condition of constant speed of light.

which can be decomposed into h– and v–components using spliting of type $\mathbf{e}_\alpha{}^{\alpha'} = \left(e_i{}^{i'}, e_a{}^{a'}\right), \mathbf{e}_\beta = (\mathbf{e}_j, e_b)$ and $\widehat{\mathbf{\Gamma}}^\gamma{}_{\alpha\beta} = (\widehat{L}^i_{jk}, \widehat{L}^a_{bk}, \widehat{C}^i_{jc}, \widehat{C}^a_{bc})$. The values $\widehat{\mathbf{\Gamma}}^{\alpha'}{}_{\beta\nu'}$ are analogs of spin connection coefficients $\Gamma^{\alpha'}{}_{\beta\nu'}$ in general relativity; $\widehat{\mathbf{\Gamma}}^{\alpha'}{}_{\beta\nu'|Z\to 0} \to \Gamma^{\alpha'}{}_{\beta\nu'}$.

The N–adapted parallel transport of a d–vector $v^{\alpha'}$ with respect to an orthonormalized N–frame $\mathbf{e}_{\beta'}$ is defined by line elements

$$\delta v^{\alpha'} = -\widehat{\mathbf{\Gamma}}^{\alpha'}{}_{\beta\gamma'}(u^\beta)v^{\gamma'}\delta u^\beta, \quad \text{where} \quad \delta u^\beta = \mathbf{e}^\beta{}_{\beta'}(u^\alpha)v^{\beta'}\delta\tau.$$

We can say that the change of spacial components (labeled by small Greek indices with hats, for a 3–d velocity space which in the relativistic case is subjected to the condition of hyperbolicity (22)) of the velocity vector d–field $v^{\widehat{\alpha}'}$ is driven by a formal (gravitational) force $\widehat{\mathbf{\Gamma}}^{\widehat{\alpha}'}{}_{\beta\widehat{\gamma}'}(u^\beta)$. One could be additional contributions from a "stochastic force" (such a noise can from any classical or quantum gravitational, or matter fields, fluctuations) associated to a Wiener process $\delta\mathcal{W}^{\widehat{\alpha}'}$. For relativistic constructions, we have to consider stochastic processes along the orthonormal frames $E^{\widehat{\alpha}'}_{\widehat{\alpha}}(v^\beta)$ as we considered in (25). In this subsection, the position space is not the Minkovski spacetime ${}^3_1M$ but a nonholonomic (pseudo) Riemannian one $\mathbf{V}$; we can generalize the constructions considering that in any point $\mathbf{u} \in \mathbf{V}$ it is defined a hyperbolic velocity space with the "fiber" metric and connections determined by

$$h_{\widehat{\alpha}\widehat{\beta}}(v^\gamma) = \delta_{\widehat{\alpha}\widehat{\beta}} - v^{\widehat{\alpha}}v^{\widehat{\beta}}/(v^3)^2, \ \gamma^{\widehat{\alpha}}_{\widehat{\beta}\widehat{\gamma}}(v^\varphi) = v^{\widehat{\alpha}}h_{\widehat{\alpha}\widehat{\beta}}$$

as in (23) but with that difference that in this subsection indices run different values (for instance, $\widehat{\alpha} = 1, 2, 4$) than those in used in subsection 3.1.

Considering that on a nonholonomic (pseudo) Riemannian manifold the Wiener process is moved along the orthonormalized frames $E^{\widehat{\alpha}'}_{\widehat{\alpha}}(v^\beta)$ in the 3–d hyperbolic velocity space on $u^\beta \in \mathbf{V}$, when (with summation on repeating indices)

$$E^{\widehat{\alpha}}_{\widehat{\alpha}'}E^{\widehat{\beta}}_{\widehat{\alpha}'} = h^{\widehat{\alpha}\widehat{\beta}}, \ \text{equivalently} \ h_{\widehat{\alpha}\widehat{\beta}}E^{\widehat{\alpha}}_{\widehat{\alpha}'}E^{\widehat{\beta}}_{\widehat{\beta}'} = \delta_{\widehat{\alpha}'\widehat{\beta}'}, \tag{31}$$

where $h^{\widehat{\alpha}\widehat{\beta}}$ is inverse to the hyperbolic metric $h_{\widehat{\alpha}\widehat{\beta}}$, the infinitesimal motion of the velocity $v^{\widehat{\alpha}}$ is determined by equations

$$\delta v^{\widehat{\alpha}} = E^{\widehat{\alpha}}_{\widehat{\alpha}'}\delta v^{\widehat{\alpha}'} \ \text{and} \ \delta E^{\widehat{\alpha}}_{\widehat{\alpha}'}(\tau) = -\gamma^{\widehat{\alpha}}_{\widehat{\beta}\widehat{\gamma}}(v^{\widehat{\varphi}})E^{\widehat{\gamma}}_{\widehat{\alpha}'}\delta v^{\widehat{\beta}}. \tag{32}$$

A random N–adapted curve on $\mathbf{V}$, parametrized in the phase–space can be introduced as a Wiener process: $dv^{\widehat{\alpha}'} \to \delta\mathcal{W}^{\widehat{\alpha}'}$. We can consider a formal noise force ${}^{noise}B^{\widehat{\alpha}} = E^{\widehat{\alpha}}_{\widehat{\alpha}'}(\tau) \circ \delta\mathcal{W}^{\widehat{\alpha}'}$. Following constraints (30) and algebraic relations (31), we conclude that the relativistic diffusion on spacetime $\mathbf{V}$ is defined by a more restricted system of coefficients because there are admitted only such $E^{\widehat{\alpha}}_{\widehat{\alpha}'}(\tau)$ when a direct relation to the hyperbolic geometry is established. We can say that we model via $\widehat{\mathbf{\Gamma}}^{\alpha'}{}_{\beta\nu'}$ a consistent description of Markovian diffusion in general relativity but keeping certain "geometric memory" on nonholonomic h-v–splitting and spacetime (pseudo) Riemannian geometry, all encoded into a corresponding Stratonovich relativistic calculus.

The stochastic differential equations describing the N–adapted relativistic diffusion of gravitational and external force fields[15] are written

$$
\begin{aligned}
\delta u^\alpha &= e^\alpha{}_{\alpha'}(u^\beta)v^{\alpha'}\delta\tau, \\
\delta v^{\widehat{\alpha}} &= E^{\widehat{\alpha}}_{\widehat{\alpha}'}(\tau) \circ \delta\mathcal{W}^{\widehat{\alpha}'} - \widehat{\mathbf{\Gamma}}^{\widehat{\alpha}}{}_{\beta\gamma'}(u^\gamma)e^\beta{}_{\alpha'}(u^\gamma)v^{\gamma'}v^{\alpha'}\delta\tau + {}^{ex}B^{\widehat{\alpha}}\delta\tau \\
\delta E^{\widehat{\alpha}}_{\widehat{\alpha}'}(\tau) &= -\gamma^{\widehat{\alpha}}_{\widehat{\beta}\widehat{\gamma}}(v^{\widehat{\varphi}})E^{\widehat{\gamma}}_{\widehat{\alpha}'} \circ \delta v^{\widehat{\beta}},
\end{aligned}
\tag{33}
$$

where a possible additional external force ${}^{ex}B^{\widehat{\alpha}} = F^{\widehat{a}}/m_0$ is defined by spacial components of a 4–force F^a acting on particles of rest mass m_0 and τ is a parameter defined along of world line of particles (in our case, moving with nonholonomic constraints on $\mathbf{V}$); we can consider τ as the phase–space proper time.

The system of stochastic equations (32) and (33) is a respective analogous of (27) and (28). There are two substantial difference between such systems of equations: the first one is for general relativity when the N–connection structure is defined for the base spacetime manifold but the second one is for special relativity nonholonomically extended as a stochastic geometric model for $\mathbf{N}$: $TT({}^3_1M) = hT({}^3_1M) \oplus vT({}^3_1M)$. Finally, we note that, for instance, there are satisfied sufficient and necessary conditions for the existence and uniqueness of N–adapted relativistic stochastic differential equations (33) if the drift and diffusion coefficients are subjected to uniform Lipschiz conditions (see details in [23, 24, 25]) and the stochastic process $\mathbf{X}(\tau) = \{\mathbf{u}(\tau), \mathbf{v}(\tau)\}$ is N–adapted to the Wiener process $\mathcal{W}^{\widehat{\alpha}'}(\tau)$, when the output $\mathbf{X}({}^2\tau)$ is a function of $\mathcal{W}^{\widehat{\alpha}'}({}^1\tau)$ up to that time, for ${}^1\tau \leq {}^2\tau$.

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[15] i.e. the nonholonomic Langevin equations in general relativity

3.2.2 Diffusion and N–adapted stochastic GR processes

A model of general relativistic of $(\mathbf{A}, \mathbf{L})$–diffusion on the fiber bundle $\mathbf{F(V)}$ with local coordinates $^{E}\mathbf{r} = \{u^{\alpha} = (x^{i}, y^{a}), v^{\widehat{\beta}}, E^{\widehat{\alpha}}_{\widehat{\alpha}'}\}$ can be derived from (32) and (33). The N–adapted diffusion operator $^{\mathbf{F(V)}}\mathbf{A}$ can be construction similarly to (17) by using operators

$$^{F}L_{\widehat{a}'} = E^{\widehat{\alpha}}_{\widehat{\alpha}'} \frac{\partial}{\partial v^{\widehat{\alpha}}} - \gamma^{\widehat{\alpha}}_{\widehat{\varepsilon}\widehat{\gamma}}(v^{\widehat{\varphi}}) E^{\widehat{\gamma}}_{\widehat{\alpha}'} E^{\widehat{\varepsilon}}_{\widehat{\beta}'} \frac{\partial}{\partial E^{\widehat{\alpha}}_{\widehat{\beta}'}},$$

$$^{F}\mathbf{L}_{0} = \mathbf{e}^{\alpha}_{\alpha'}(u^{\beta}) v^{\alpha'} \mathbf{e}_{\alpha} - \widehat{\mathbf{\Gamma}}^{\widehat{\alpha}}_{\beta\gamma'}(u^{\gamma}) \mathbf{e}^{\beta}_{\alpha'}(u^{\gamma}) v^{\gamma'} v^{\alpha'} \frac{\partial}{\partial v^{\widehat{\alpha}}} + {}^{ex}B^{\widehat{\alpha}} \frac{\partial}{\partial v^{\widehat{\alpha}}} - \gamma^{\widehat{\alpha}}_{\widehat{\varepsilon}\widehat{\gamma}}(v^{\widehat{\varphi}}) E^{\widehat{\gamma}}_{\widehat{\beta}'} B^{\widehat{\varepsilon}} \frac{\partial}{\partial E^{\widehat{\alpha}}_{\widehat{\beta}'}},$$

where the force $B^{\widehat{\varepsilon}}$ consists respectively from the gravitational and external components,

$$B^{\widehat{\varepsilon}} = -\widehat{\mathbf{\Gamma}}^{\widehat{\varepsilon}}_{\beta\gamma'}(u^{\gamma}) e^{\beta}_{\alpha'}(u^{\gamma}) v^{\gamma'} v^{\alpha'} + {}^{ex}B^{\widehat{\varepsilon}}.$$

For real applications, it is convenient to use the operator $^{\mathbf{P}}\mathbf{A}$ as the projection of $^{\mathbf{F(V)}}\mathbf{A}$ on the phase space with coordinates $\mathbf{r} = \{u^{\alpha} = (x^{i}, y^{a}), v^{\widehat{\beta}}\}$ when for corresponding functions $\mathbf{f}$ and f the condition $^{\mathbf{F(V)}}\mathbf{A}\mathbf{f}(^{E}\mathbf{r}) = {}^{\mathbf{P}}\mathbf{A}f(\mathbf{u}, \mathbf{v})$. The N–adapted diffusion operator in the phase space is given by

$$^{\mathbf{P}}\mathbf{A} = {}^{v}\triangle + \mathbf{e}^{\alpha}_{\alpha'}(u^{\beta}) v^{\alpha'} \mathbf{e}_{\alpha} + B^{\widehat{\alpha}} \frac{\partial}{\partial v^{\widehat{\alpha}}},$$

where the Laplace–Beltrami operator in the hyperbolic velocity space is

$$^{v}\triangle = \delta^{\widehat{\alpha}'\widehat{\beta}'} E^{\widehat{\alpha}}_{\widehat{\alpha}'} \frac{\partial}{\partial v^{\widehat{\alpha}}} E^{\widehat{\beta}}_{\widehat{\beta}'} \frac{\partial}{\partial v^{\widehat{\beta}}}$$

$$= h^{\widehat{\alpha}\widehat{\beta}} \left(\frac{\partial^{2}}{\partial v^{\widehat{\alpha}} \partial v^{\widehat{\beta}}} + \gamma^{\widehat{\varepsilon}}_{\widehat{\alpha}\widehat{\beta}} \frac{\partial}{\partial v^{\widehat{\varepsilon}}} \right) = \frac{1}{\sqrt{|h_{\widehat{\alpha}\widehat{\beta}}|}} \frac{\partial}{\partial v^{\widehat{\varepsilon}}} \left(\sqrt{|h_{\widehat{\alpha}\widehat{\beta}}|} h^{\widehat{\varepsilon}\widehat{\mu}} \frac{\partial}{\partial v^{\widehat{\mu}}} \right).$$

This operator is self–adjoint, $^{v}\triangle = ({}^{v}\triangle)^{+}$.

The corresponding backward Kolmogorov equation for the N–adapted general relativistic stochastic processes (33) is written in the form

$$\frac{\delta}{\delta\tau} \varphi(\tau, \mathbf{u}, \mathbf{v}) = {}^{\mathbf{P}}\mathbf{A} \varphi(\tau, \mathbf{u}, \mathbf{v}).$$

Using the adjoint of d–operator $^{\mathbf{P}}\mathbf{A}$, it is possible to construct the corresponding Fokker–Planck equation in phase space (this equation in general relativity is also called the Kramer equation). Introducing the probability density function $\Phi := \varphi(\tau, \mathbf{u}, \mathbf{v})$ (as the transition probability $\Phi(\mathbf{u}, \mathbf{v}, \tau | \mathbf{u}_{0}, \mathbf{v}_{0}, \tau = 0)$), we write the Fokker–Planck N–adapted equation in general relativity on a nonholonomic spacetime with d–metric $\mathbf{g}_{\alpha\beta}$ (2) and 3–d hyperbolic metric $h_{\widehat{\alpha}\widehat{\beta}}$:

$$\frac{\delta\Phi}{\partial\tau} = -\frac{v^{\alpha'}}{\sqrt{|\mathbf{g}_{\alpha\beta}|}} \mathbf{e}_{\gamma} \left(\sqrt{|\mathbf{g}_{\alpha\beta}|} \mathbf{e}^{\gamma}_{\alpha'}(u^{\beta}) \Phi \right) - \frac{1}{\sqrt{|h_{\widehat{\alpha}\widehat{\beta}}|}} \frac{\partial}{\partial v^{\widehat{\varepsilon}}} \left(\sqrt{|h_{\widehat{\alpha}\widehat{\beta}}|} B^{\widehat{\alpha}} \Phi \right) + \frac{\rho}{2} {}^{v}\triangle\Phi, \tag{34}$$

where the first two terms in the right side are the divergence d–operators in, respectively, the position and velocity spaces.

We use the phase–space proper time τ in the N–adapted general diffusion equation (33) (see detailed explanations in [24] how nontrivial torsion terms can be included in the drift coefficients with additional terms in $B^{\widehat{\alpha}}$). Alternatively, we can consider parametrizations in terms of coordinate time (in this work, $u^{3} = t$) which is convenient for introducing gravitational and external forces fields. In such cases, the observer time with N–adapted infinitesimal element $\delta u^{3} = \delta y^{3} = \mathbf{e}^{3}_{\alpha'}(u^{\beta}) v^{\alpha'} \delta\tau$ is a function of the proper time τ and the space and velocity variables. A simple analogy with non–relativistic diffusion formula is possible for such frames of references when $\mathbf{e}^{3}_{\widehat{\alpha}'} = 0$ which can be introduced in N–adapted Arnovitt–Deser–Misner (ADM). We can introduce $\delta u^{3} = \delta y^{3} = \sqrt{|h_{33}(u^{\beta})|} v^{3} \delta\tau$, where $v^{3} = \left[1 + (v^{1})^{2} + (v^{2})^{2} + (v^{4})^{2} \right]^{1/2}$. Such a model of general relativistic diffusion in ADM variables, for trivial N–splitting is elaborated in [8] (see formulas (36)–(46) in that work). To generate exact stochastic solutions of Einstein equations we have to consider nontrivial N–connection structures which makes the stochastic/diffusion theory more complex but allows us to separate and integrate the fundamental field equations.

4 Exact Stochastic Solutions in Gravity

The anholonomic deformation/frame method of constructing exact solutions in gravity [17,18,21] can be extended to a formalism with stochastic processes and diffusion. In this section, we provide necessary geometric preliminaries on gravitational field equations on nonholonomic (pseudo) Riemannian manifolds, show how such equations can be formally integrated in very general forms and provide some general criteria/conditions when certain components of metrics and connections are induced by stochastic generating functions and corresponding diffusion processes.

4.1 The Einstein equations on nonholonomic manifolds

In standard form, the Einstein equations on a nonholonomic (pseudo) Riemannian manifold (spacetime) $\mathbf{V}$ are written in terms of the Ricci tensor, $R_{\beta\delta}$, and scalar curvature, R, for the Levi–Civita connection ∇, for a given source, i.e. energy–momentum tensor for matter, $T_{\alpha\beta}$,[16]

$$R_{\beta\delta} - \frac{1}{2}g_{\beta\delta}R = \varkappa T_{\beta\delta}, \tag{35}$$

where $\varkappa = const$. It is not possible to integrate analytically, in general form, this system of partial differential equations because of it generic nonlinearity and complexity. In the above mentioned works (see also references therein), we proved in details that very general integral varieties can be constructed if we rewrite the equations (35) in terms of, for instance, the canonical d–connection $\widehat{\mathbf{D}}$,

$$\widehat{\mathbf{R}}_{\beta\delta} - \frac{1}{2}\mathbf{g}_{\beta\delta}\,{}^sR = \mathbf{\Upsilon}_{\beta\delta}, \tag{36}$$

$$\widehat{L}^c_{aj} = e_a(N^c_j),\ \widehat{C}^i_{jb} = 0,\ \Omega^a_{ji} = 0. \tag{37}$$

In the above formulas, $\widehat{\mathbf{R}}_{\beta\delta}$ is the Ricci tensor for $\widehat{\mathbf{\Gamma}}^\gamma_{\alpha\beta}$, ${}^sR = \mathbf{g}^{\beta\delta}\widehat{\mathbf{R}}_{\beta\delta}$ and $\mathbf{\Upsilon}_{\beta\delta}$ is constructed for the same metric but with $\widehat{\mathbf{D}}$, similarly to formulas with ∇, when $\mathbf{\Upsilon}_{\beta\delta} \to \varkappa T_{\beta\delta}$ for $\widehat{\mathbf{D}} \to \nabla$. If the constraints (37) are satisfied the tensors $\widehat{\mathbf{T}}^\gamma_{\alpha\beta}$ (8) and $Z^\gamma_{\alpha\beta}$ from (9) are zero and (36) are equivalent to (35). Using nonholonomic deformations and "non–tensor" transformation laws for the coefficients of the linear and d–connections, we can satisfy the condition $\widehat{\mathbf{\Gamma}}^\gamma_{\alpha\beta} = \Gamma^\gamma_{\alpha\beta}$, with respect to N–adapted frames (5) and (6), see (9), even $\widehat{\mathbf{D}} \neq \nabla$.

Any exact solution of Einstein equations can be parametrized as a metric (3). In classical gravity the coefficients $g_{ij} = diag[g_i = \eta_i\,{}^\circ g_i]$, $h_{ab} = diag[h_a = \eta_a\,{}^\circ h_a]$ and $N^3_k = w_i = \eta^3_i\,{}^\circ w_i$, $N^4_k = n_i = \eta^4_i\,{}^\circ n_i$ are certain smooth and/or singular "non–random" classes of real functions defining, for instance, a black hole/worm hole /cosmological etc solution. The main goal of this paper is to prove that we can construct new classes of Einstein equations, generated by some stochastic/random gravitational 'polarizations' η_α and η^a_i when the nonholonomic deformation of metric ${}^\circ\mathbf{g} = [\,{}^\circ g_i,\ {}^\circ h_a,\ {}^\circ N^a_k] \to\ {}^\eta\mathbf{g} = [\,g_i, h_a, N^a_k]$ result both in solutions of equations (36) and (37), or (35), and any type of stochastic/diffusion equation (for various purposes, we can consider certain variants of relativistic diffusion, backward Kolmogorov, Fokker–Planck etc equations). For explicit constructions, we can consider that ${}^\circ\mathbf{g}$ is a (pseudo) Riemannian metric (which can be, or not, a solution of classical Einstein equations) but impose the condition that a metric ${}^\eta\mathbf{g}$ define a solution (or a class of solutions) of (36) when certain coefficients are additionally generated by some stochastic/diffusion processes in curved spaces.

Various types of stochastic generalizations of Einstein manifolds (with mixed types of non–random and random processes, mutual diffusion of gravitational and matter fields etc) are called in this work as stochastic Einstein spaces. Here we emphasize that because of generic nonlinearity of gravitational field equations the solutions may be with chaos, stochastics, fractional behavior etc even we may put certain well defined classical boundary/initial conditions on integration functions. Such nonlinear stochastic gravitational and matter field configurations are with a very complex and mixed random–sure spacetime structure and very sophisticate rules of stochastic and nonholonomic differentiation and integration.

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[16] In brief, the gravitational field equations in general relativity are defined geometrically in this form: Denoting by $\nabla = \{\Gamma^\alpha_{\beta\gamma}\}$ the Levi–Civita connection (uniquely defined by a given tensor $\mathbf{g}$ to be metric compatible, $\nabla\mathbf{g} = 0$, and with zero torsion), the coefficients of necessary tensors are computed with respect to an arbitrary local frame basis $e_\alpha = (e_i, e_a)$ and its dual basis $e^\beta = (e^j, e^b)$. Using the Riemannian curvature tensor $\mathcal{R} = \{R^\alpha_{\beta\gamma\delta}\}$ of ∇, we define the Ricci tensor, $\mathcal{R}ic = \{R_{\beta\delta} \doteq R^\alpha_{\beta\alpha\delta}\}$, compute the scalar curvature $R \doteq g^{\beta\delta}R_{\beta\delta}$, where $g^{\beta\delta}$ is inverse to $g_{\alpha\beta}$.

4.2 Generating stochastic solutions of Einstein equations

For an ansatz of type (3), the Einstein equations (36) for $\widehat{\mathbf{D}}$ with a general source of type[17] $\Upsilon^{\alpha}{}_{\beta} = diag[\Upsilon_{\gamma}; \Upsilon_1 = \Upsilon_2 = \Upsilon_2(x^k, v); \Upsilon_3 = \Upsilon_4 = \Upsilon_4(x^k)]$ transform into a system of nonlinear partial differential equations with separation of equations for h– and v–components of metric and N–connection coefficients,

$$\widehat{R}_1^1 = \widehat{R}_2^2 = \frac{-1}{2g_1 g_2}[g_2^{\bullet\bullet} - \frac{g_1^{\bullet} g_2^{\bullet}}{2g_1} - \frac{(g_2^{\bullet})^2}{2g_2} + g_1'' - \frac{g_1' g_2'}{2g_2} - \frac{(g_1')^2}{2g_1}] = -\Upsilon_4(x^k), \tag{38}$$

$$\widehat{R}_3^3 = \widehat{R}_4^4 = -\frac{1}{2h_3 h_4}[h_4^{**} - \frac{(h_4^*)^2}{2h_4} - \frac{h_3^* h_4^*}{2h_3}] = -\Upsilon_2(x^k, v), \tag{39}$$

$$\widehat{R}_{3k} = \frac{w_k}{2h_4}[h_4^{**} - \frac{(h_4^*)^2}{2h_4} - \frac{h_3^* h_4^*}{2h_3}] + \frac{h_4^*}{4h_4}\left(\frac{\partial_k h_3}{h_3} + \frac{\partial_k h_4}{h_4}\right) - \frac{\partial_k h_4^*}{2h_4} = 0, \tag{40}$$

$$\widehat{R}_{4k} = \frac{h_4}{2h_3}n_k^{**} + \left(\frac{h_4}{h_3}h_3^* - \frac{3}{2}h_4^*\right)\frac{n_k^*}{2h_3} = 0, \tag{41}$$

$$w_i^* = \mathbf{e}_i \ln|h_4|, \mathbf{e}_k w_i = \mathbf{e}_i w_k, \ n_i^* = 0, \ \partial_i n_k = \partial_k n_i \tag{42}$$

In brief, we wrote the partial derivatives in the form $a^{\bullet} = \partial a/\partial x^1$, $a' = \partial a/\partial x^2$, $a^* = \partial a/\partial v$. The ansatz (3) and resulting system of equations does not depend on variable y^4 (we do not have terms with $\partial/\partial y^4$, i.e. our original ansatz was taken with one Killing symmetry; see references [17,18] how to construct exact solutions with general "non–Killing" symmetries). The constraints (42) have to be imposed additionally if we wont to satisfy the conditions (37) and generate solutions in general relativity just for the Levi–Civita connection ∇.

The above system of equations can be integrated in very general forms [17,18], in sure variables, by integrating step by step the equations beginning with (38), which independent from another ones (relating two coefficients of h–metric, g_1 and g_2, and source Υ_4, all depending on two variables x^k), then (39) (relating two coefficients of v–metric, h_3 and h_4, and source Υ_2, all depending on three variables x^k, v, where we take $v = t$; the time coordinate can be related for ansatz of type (3) with the parameter τ via relation $\delta\tau = \sqrt{|h_3|}\delta t$). The equations (40) is an algebraic one for N–connection coefficients w_k and we have to integrate two times the equations (41) in order to compute the N–connection coefficients n_k; in both cases, we can generate sure or stochastic solutions, depending on the type of solutions we have for h_3 and/or h_4. Finally, the equations (42) impose additional constraints on integral varieties for (38)–(41) eliminating induced (by off–diagonal coefficients of metric, i.e. N–connection) torsion and constraining the system of equations and solutions to be just for ∇.

The h– and v–separation allows us not only to generalize the constructions for stochastic metrics and N–connections but also to analyze the conditions when a non–random h–metric structure model gravitational diffusion processes in the v–subspace.

4.3 Stochastic solutions with $h_{3,4}^* \neq 0$ and $\Upsilon_{2,4} \neq 0$

We consider a metric (3) when

$$^{\eta}\mathbf{g} = e^{\psi(x^k)}dx^i \otimes dx^i + h_3(x^k, t)\mathbf{e}^3 \otimes \mathbf{e}^3 + h_4(x^k, t)\mathbf{e}^4 \otimes \mathbf{e}^4, \tag{43}$$
$$\mathbf{e}^3 = dt + w_i(x^k, t)dx^i, \mathbf{e}^4 = dy^4 + n_i(x^k, t)dx^i$$

is supposed to be a solution of (38)–(41) with $g_1 = g_2 = e^{\psi(x^k)}$ defining a non–random h–metric, as a solution of 2-d Laplace equation (the h–components of the Einstein equations transform to this simple equation which can be used for generating sure, or random solutions)

$$\ddot{\psi} + \psi'' = 2\Upsilon_4(x^k), \tag{44}$$

and h_3, h_4 defining a stochastic v–metric (in result, w_i and/or n_i can be also stochastic solutions).

Introducing values

$$\phi = \ln|\frac{h_4^*}{\sqrt{|h_3 h_4|}}|, \ \alpha_i = h_4^* \partial_i \phi, \ \beta = h_4^* \ \phi^*, \ \gamma = \left(\ln|h_4|^{3/2}/|h_3|\right)^*, \tag{45}$$

---

[17] parametrization of energy–momentum tensors in the above presented form are possible by corresponding nonholonomic frame and/or coordinate frame transform for various types of matter sources, including some general and important cases with cosmological constants and various models of locally anisotropic fluid/scalar field/ spinor/ gauge fields interactions on curved spaces

the equations (40),(41) are respectively written in the form

$$\beta w_i + \alpha_i = 0, \tag{46}$$
$$n_i^{**} + \gamma n_i^* = 0 \tag{47}$$

The type of solutions for N–connection coefficients depend explicitly on the type of solutions we construct/choose for the v–metric.

4.3.1 Non–random/sure solutions

For sure coefficients, the equation (39) transform into

$$h_4^* = 2h_3 h_4 \Upsilon_2(x^i, t)/\phi^*. \tag{48}$$

If $h_4^* \neq 0; \Upsilon_2 \neq 0$, we get $\phi^* \neq 0$. Prescribing any non–constant $\phi = \phi(x^i, t)$ as a generating function, we can construct exact solutions of (48)–(47). Integrating on t, in order to determine h_3, h_4 and n_i, and solving algebraic equations, for w_i, we get

$$h_3 = \pm \frac{|\phi^*(x^i, t)|}{\Upsilon_2}, \ \ h_4 = \ ^0h_4(x^k) \pm \ 2 \int \frac{(\exp[2 \ \phi(x^k, t)])^*}{\Upsilon_2} dt, \tag{49}$$

$$w_i = -\partial_i \phi/\phi^*, \ \ n_i = \ ^1n_k\left(x^i\right) + \ ^2n_k\left(x^i\right) \int [h_3/(\sqrt{|h_4|})^3] dt,$$

where $\ ^0h_4(x^k)$, $\ ^1n_k\left(x^i\right)$ and $\ ^2n_k\left(x^i\right)$ are integration functions. We have to fix a corresponding sign $\pm$ in order to generate a necessary local signature of type $(+ + -+)$ for some chosen ϕ, Υ_2 and Υ_4. Such solutions include as particular cases the classes of solutions for a nontrivial cosmological constant $\Upsilon_i = \lambda$, or nonholonomic configurations with polarizations of such constants, $\lambda \to \ ^h\lambda(x^k) = \Upsilon_4(x^k)$ and $\lambda \to \ ^v\lambda(x^k, t) = \Upsilon_2(x^k, t)$.

In Refs. [17,18,21], we studied various classes of parametric and non–parametric exact solutions with coefficients of type (44) and (48) describing black ellipsoids, locally anisotropic wormholes and cosmological solutions and their noncommutative generalizations. Those constructions can be generalized for stochastic Einstein spaces if ϕ and/or Υ_2 are taken to be certain stochastic functions.

4.3.2 Metrics with h–diffusion of generating functions

This class of solutions if we chose, for instance, a generating function

$$\phi(x^k, t) \to \underline{\phi}(x^k, t) = \phi(x^k, t) + \varpi \widetilde{\phi}(x^k, t), \tag{50}$$

where $\widetilde{\phi}(x^k, t) \sim \underline{f}(\tau, \mathbf{u}) = \underline{f}(\tau, x^i)$ is defined as as diffusion process from h–space to v–space by a corresponding backward Kolmogorov equation for the N–adapted general relativistic stochastic processes (20) and/or (21) (in order to state the solutions in exact form, we consider only h–operators and fix $\mathbf{A}^\beta = 0$ in $\ ^{\mathbf{V}}\widetilde{\mathbf{A}}$ (18)), where $\tau \to t$, $\mathbf{v} = \mathbf{v}_0 = const$ and the Laplace–Beltrami d–operator $\widehat{\triangle}$ (19) is computed as $\widehat{\triangle} = h\widehat{\triangle}$ for $g_{ij} = \delta_{ij} e^{\psi(x^k)}$. We can consider ϖ as a real parameter which for $\varpi = 0$ transforms our system into a non–random one. For solutions with one Killing symmetry such parameters can be always introduced. Nevertheless, there are substantial differences between the parameter ϖ and those for sure solutions with superposition of Killing symmetries and/or with noncommutative parameters θ, see [21]. In this work, a nontrivial ϖ results in stochastic behavior of (some) coefficients of metrics and connections, i.e. of gravitational fields.

The generalized Kolmogorov backward equation (describing diffusion of gravitational h–components on τ, i. e. into v–components) is

$$\partial_\tau \underline{f}(\tau, x^i) = \ h\widehat{\triangle}\underline{f}((\tau, x^i) = h\widehat{\triangle}\underline{f}((\tau, x^i), \tag{51}$$
$$\underline{f}(0, x^i) = f(x^i).$$

The corresponding generalized Fokker–Planck equation is

$$\partial_\tau F = \frac{\rho}{2} h\widehat{\triangle}F, \tag{52}$$

where $\mathcal{F} = \mathcal{F}(\tau, {}^1x^i; 0, {}^2x^i)$ is the transition probability with the initial condition $\mathcal{F}(0, {}^1x^i; 0, {}^2x^i) = \delta({}^1x^i - {}^2x^i)$ for any two points ${}^1x^i, {}^2x^i \in \mathbf{V}$ and adequate boundary conditions at infinity. The same equations can be considered for the probability density $\varphi(\tau, x^i)$ but for the initial condition $\varphi(\tau = 0, x^i) = {}^0\varphi(x^i)$. Stochastic components $\widetilde{\phi}(x^k, t)$ in (49) and (50) generated as diffusion solutions of (51) and/or (52) provide some "simple" and explicit examples of stochastic metrics in Einstein gravity, all distinguished by corresponding nonholonomic constraints. Under general frame/coordinate transforms, the parametizatons change into mixed holonomic/nonholonomic and non–random/stochastic variables both on h– and v–subspaces. More general types of nonholonomic diffusion equations can be also considered but the solutions will be generated in non–explicit forms.

Fixing ${}^2n_k(x^i) = 0$ in (49), we generate non–random solutions for the N–connection components n_i. But w_i will have positively a stochastic character, as well both h_3 and h_4, if the generating function $\underline{\phi(x^k, t)}$ contains a nontrivial diffusion component. Such random gravitational processes can be induced even as vacuum gravitational configurations for smooth coefficients of Einstein equations

4.3.3 Solutions induced by random sources

There is an alternative possibility to generate random gravitational processes than those with stochastic generating functions. This can be seen from formulas (48) and (49) for certain random/noise behavior of source Υ_2. Such constructions are similar to those for *stochastic gravity* (based on the Einstein–Langeven equation with additional sources due to the noise kernel, see a review and references in [26]). Nevertheless, it should be mentioned here that our anholonomic deformation method is different from the methods of deriving solutions in stochastic gravity. Via nonholonomic transforms/deformations we can generate in a "non–perturbative" manner different classes of exact solutions with generic off–diagonal terms and mixed holonomic–nonholonomic variables and matter sources.

In a particular case, we can consider that a stochastic Υ_2 can be constructed as a random vacuum polarization of gravitational constant, when vacuum gravitational fields are nonholonomically distorted as a diffusion process, for instance, modelled by solutions of equations (51) and/or (52) associated to some effective matter fiels/cosmological constant.

4.3.4 Diffusion to the Levi–Civita conditions

In order to construct exact solutions for the Levi–Civita connection, i.e. of standard form of Einstein equations written with respect to N–adapted frames, we have to constrain the coefficients (49) of metric (43) to satisfy the conditions (42). There are various possibilities. For instance, we may consider a classical sure metric (which may be, or not, a solution of gravitational field equations with nontrivial distortion) and constrain the system in such a way that gravitational diffusion will result in stochastic vacuum or Einstein spaces. There are scenarios when a sure Einstein configuration is stochastically transformed into nonholonomic configuration with nontrivial distortion of of the Levi–Civita connection.

The above mentioned gravitational diffusion evolution models depend on the class of additional constraints we impose on generating and integration functions. For instance, we can chose that ${}^2n_k(x^i) = 0$ and ${}^1n_k(x^i)$ are any functions satisfying the non–random conditions $\partial_i\,{}^1n_k = \partial_k\,{}^1n_i$. The constraints for $\phi(x^k, t)$ can be for random, sure and/or of mixed nature variables, following from constraints on N–connection coefficients $w_i = -\partial_i\phi/\phi^*$,

$$
\begin{aligned}
(w_i[\phi])^* + w_i[\phi]\,(h_4[\phi])^* + \partial_i h_4[\phi] &= 0, \\
\partial_i\,w_k[\phi] &= \partial_k\,w_i[\phi],
\end{aligned}
\tag{53}
$$

where, for instance, we denoted by $h_4[\phi]$ the functional dependence on ϕ. So, if ϕ is stochastic, we have to consider that (53) are some relations on mathematical expectations etc. Such conditions are always satisfied for cosmological solutions with $\phi = \phi(t)$ or if $\phi = const$ (in the last case $w_i(x^k, t)$ can be any non–random and/or stochastic functions as follows from (46) with zero β and α_i, see (45)).

4.4 Special cases of stochastic Einstein spaces

We can construct such solutions for certain special parametrizations of coefficients for ansatz (49) subjected to the condition to be solutions of equations (44)–(47) and certain nonholonomic diffusion equations on curved spaces.

4.4.1 Vacuum gravitational diffusion with $h_4^* = 0$

The equation (39) can be solved for such a case, $h_4^* = 0$, only if $\Upsilon_2 = 0$. So, in N–adapted frames, the v–components of gravitational equations are vacuum ones. We can consider any functions $w_i(x^k, t)$, being sure or random ones, as solutions of (40), and its equivalent (46), because the coefficients β and α_i, see (45), are zero.

We find nontrivial values of n_i by integrating (47) for $h_4^* = 0$ and any given h_3 which results in $n_i = {}^1n_k(x^i) + {}^2n_k(x^i) \int h_3 dt$. Choosing h_3 to be a stochastic function, we have to compute $\int h_3 dt$ as a Stratonovich stochastic integral, when $\int h_3 dt \to \int \widetilde{h}_3 \circ d\tau$. We can consider any $g_1 = g_2 = e^{\psi(x^k)}$, with $\psi(x^k)$ determined by (44) for a given $\Upsilon_4(x^k)$; for simplicity, we can consider only non–random solutions for the h–metric.

Summarizing the constructions, we get a class of stochastic gravitational solutions defined by ansatz

$$
{}^\eta\mathbf{g} = e^{\psi(x^k)}dx^i \otimes dx^i + h_3(x^k, t)\mathbf{e}^3 \otimes \mathbf{e}^3 + {}^0h_4(x^k)\mathbf{e}^4 \otimes \mathbf{e}^4, \tag{54}
$$
$$
\mathbf{e}^3 = dt + w_i(x^k, t)dx^i,
$$
$$
\mathbf{e}^4 = dy^4 + [\ {}^1n_k(x^i) + {}^2n_k(x^i)\int h_3 dt \to \int \widetilde{h}_3 \circ d\tau]dx^k,
$$

for arbitrary generating stochastic and/or sure functions $h_3(x^k, t)$, $w_i(x^k, t)$, ${}^0h_4(x^k)$ and integration functions ${}^1n_k(x^i)$ and ${}^2n_k(x^i)$.

The conditions (42) selecting from (54) a subclass of solutions for the Levi–Civita connection transform into the equations

$$
{}^2n_k(x^i) = 0 \quad \text{and} \quad \partial_i\ {}^1n_k = \partial_k\ {}^1n_i,
$$
$$
w_i^* + \partial_i\ {}^0h_4 = 0 \quad \text{and} \quad \partial_i\ w_k = \partial_k\ w_i,
$$

for any such $w_i(x^k, t)$ and ${}^0h_4(x^k)$. This class of constraints do not involve the generating function $h_3(x^k, t)$. So, we can consider sure constraints to the Levi–Civita configurations for a stochastic h_3. In general, we can model nontrivial Levi–Civita diffusion processes for sure and/or stochastic h_3, but constraining the gravitational diffusion process to conditions $h_4^* = 0$ and $\Upsilon_2 = 0$.

4.4.2 Gravitational diffusion with $h_3^* = 0$ and $h_4^* \neq 0$

Such stochastic spacetimes are defined by ansatz of type

$$
{}^\eta\mathbf{g} = e^{\psi(x^k)}dx^i \otimes dx^i - {}^0h_3(x^k)\mathbf{e}^3 \otimes \mathbf{e}^3 + h_4(x^k, t)\mathbf{e}^4 \otimes \mathbf{e}^4,
$$
$$
\mathbf{e}^3 = dt + w_i(x^k, t)dx^i, \mathbf{e}^4 = dy^4 + n_i(x^k, t)dx^i, \tag{55}
$$

where $g_1 = g_2 = e^{\psi(x^k)}$, with $\psi(x^k)$ being a solution of (44) for any given $\Upsilon_4(x^k)$. The function $h_4(x^k, t)$ must satisfy the equation (48) which for $h_3^* = 0$ is just

$$
h_4^{**} - \frac{(h_4^*)^2}{2h_4} - 2\ {}^0h_3 h_4 \Upsilon_2(x^k, t) = 0.
$$

The N–connection coefficients are

$$
w_i = -\partial_i\widetilde{\phi}/\widetilde{\phi}^*,\ n_i = {}^1n_k(x^i) + {}^2n_k(x^i)\int [1/(\sqrt{|h_4|})^3]dt,
$$

when $\widetilde{\phi} = \ln|h_4^*/\sqrt{|\ {}^0h_3 h_4|}|$. Fixing ${}^0h_3 = 0$, we can always eliminate possible stochastic contributions from Υ_2 with a sure value h_4 being a solution of $h_4^{**} = (h_4^*)^2/2h_4$. This impose us to consider classical non–random values for $\widetilde{\phi}$ and, as consequences, for w_i and n_i.

If we consider any ${}^0h_3 \neq 0$, the stochastic gravitational processes will be induced by "noise" in Υ_2. So, the stochastic metrics of type (55) are generically defined by a stochastic source Υ_2. Such nonholonomic stochastic gravitational configurations are very different by those described by ansatz (54) when the gravitational diffusion is of generic vacuum type.

The Levi–Civita configurations for (55) are selected by the conditions (42) which, for this case, are satisfied if

$$
{}^2n_k(x^i) = 0\ \text{ and } \partial_i\ {}^1n_k = \partial_k\ {}^1n_i,
$$
$$
\text{and}\quad \left(w_i[\widetilde{\phi}]\right)^* + w_i[\widetilde{\phi}]\left(h_4[\widetilde{\phi}]\right)^* + \partial_i h_4[\widetilde{\phi}] = 0,
$$
$$
\partial_i\ w_k[\widetilde{\phi}] = \partial_k\ w_i[\widetilde{\phi}].
$$

Such conditions are similar to (53) but for a different relation of v–coefficients of metric to another type of generating function $\widetilde{\phi}$ which can be stochastic only for random values of Υ_2. They are always satisfied for cosmological solutions with $\widetilde{\phi} = \widetilde{\phi}(t)$ or if $\widetilde{\phi} = const$. In the last case $w_i(x^k, t)$ can be any stochastic or sure functions as follows from (46) with zero β and α_i, see (45). So, for some special configurations, random values of w_k can be induced via nonholonomic vacuum gravitational diffusion even the source Υ_2 is constrained to be non–random.

4.4.3 Solutions with constant generating functions

Fixing $\phi = \phi_0 = const$ in (45), with $h_3^* \neq 0$ and $h_4^* \neq 0$, we can express the general solutions of (44)–(47) in the form

$$
\begin{aligned}
{}^{\eta}\mathbf{g} &= e^{\psi(x^k)} dx^i \otimes dx^i - {}^0h^2 \left[f^*(x^i, t) \right]^2 |\varsigma_{\Upsilon}(x^i, t)| \mathbf{e}^3 \otimes \mathbf{e}^3 + f^2(x^i, t) \mathbf{e}^4 \otimes \mathbf{e}^4, \\
\mathbf{e}^3 &= dt + w_i(x^k, t) dx^i, \ \mathbf{e}^4 = dy^4 + n_k(x^i, t) dx^i,
\end{aligned}
\tag{56}
$$

where ${}^0h = const$, $g_1 = g_2 = e^{\psi(x^k)}$, with $\psi(x^k)$ being a solution of (44) for any given $\Upsilon_4(x^k)$, and

$$
\varsigma_{\Upsilon}(x^i, t) = \varsigma_{4[0]}(x^i) - \frac{h_0^2}{16} \int \Upsilon_2(x^k, t)[f^2(x^i, t)]^2 dt.
$$

In a stochastic sense, we should consider $\int ...dt \rightarrow \int Stratonovich \ \delta\tau$.

The N–connection coefficients $N_i^3 = w_i(x^k, t) \ N_i^4 = n_i(x^k, t)$ are

$$
w_i = -\frac{\partial_i \varsigma_{\Upsilon}(x^k, t)}{\varsigma_{\Upsilon}^*(x^k, t)}
\tag{57}
$$

$$
n_k = {}^1n_k(x^i) + {}^2n_k(x^i) \int \frac{[f^*(x^i, t)]^2}{[f(x^i, t)]^2} \varsigma_{\Upsilon}(x^i, t) \, dt,
\tag{58}
$$

with necessary generalizations to Stratonovich stochastic integrals.

We must take $\varsigma_{4[0]}(x^i) = \pm 1$ if $\varsigma_{\Upsilon}(x^i, t) = \pm 1$ for $\Upsilon_2 \rightarrow 0$. In such a case, the functions $h_3 = - {}^0h^2 \left[f^*(x^i, t) \right]^2$ and $h_4 = f^2(x^i, t)$ satisfy the equation (48) written in the form $\sqrt{|h_3|} = {}^0h(\sqrt{|h_4|})^*$, which is compatible with the condition $\phi = \phi_0$.

The gravitational diffusion for this class of solutions has two sources: the first one can be if $f(x^i, t)$ is a random generating function and the second one is for random sources $\Upsilon_2(x^k, t)$. Gravitational diffusion mix such contributions even we distinguish them nonholonomically for certain configurations.

The subclass of solutions for the Levi–Civita connection with ansatz of type (56) is selected via conditions (42). We can chose that ${}^2n_k(x^i) = 0$ and ${}^1n_k(x^i)$ are any functions satisfying the conditions $\partial_i \, {}^1n_k = \partial_k \, {}^1n_i$. The constraints on values $w_i = -\partial_i \varsigma_{\Upsilon}/\varsigma_{\Upsilon}^*$ result in constraints on ς_{Υ}, which is determined by Υ_2 and f,

$$
\begin{aligned}
(w_i[\varsigma_{\Upsilon}])^* + w_i[\varsigma_{\Upsilon}] (h_4[\varsigma_{\Upsilon}])^* + \partial_i h_4[\varsigma_{\Upsilon}] &= 0, \\
\partial_i \, w_k[\varsigma_{\Upsilon}] &= \partial_k \, w_i[\varsigma_{\Upsilon}],
\end{aligned}
\tag{59}
$$

where, for instance, we denoted by $h_4[\varsigma_{\Upsilon}]$ the functional dependence on ς_{Υ} including random and non–random contributions both from vacuum gravitational configurations and stochastic processes for matter. Such conditions are always satisfied for cosmological solutions with $f = f(t)$. For $\widehat{\mathbf{D}}$, if $\Upsilon_2 = 0$ and $\phi = const$, the coefficients $w_i(x^k, t)$ can be arbitrary functions (we can fix $\varsigma_{\Upsilon} = 1$, which does not impose a functional dependence of w_i on ς_{Υ}) as follows from (46) with zero β and α_i, see (45). As in the previous case, such N–connection components can be stochastic ones as arising from some vacuum gravitational configurations. To generate solutions for ∇ such w_i must be additionally constrained following formulas (59) re–written for $w_i[\varsigma_{\Upsilon}] \rightarrow w_i(x^k, t)$ and $h_4[\varsigma_{\Upsilon}] \rightarrow h_4(x^i, t)$. Such scenarios are typical ones with nonholonomic gravitational diffusion from/to Levi–Civita configurations.

4.4.4 Non–Killing stochastic solutions

We note that any sure and/or stochastic solution $\mathbf{g} = \{g_{\alpha'\beta'}(u^{\alpha'})\}$ of the Einstein equations (36) and/or (35) with Killing symmetry $\partial/\partial y$ (for local coordinates in the form $y^3 = t$ and $y^4 = y$) can be parametrized in a form derived in this section. Using frame transforms of type $e_\alpha = e_\alpha^{\alpha'} e_{\alpha'}$, with $\mathbf{g}_{\alpha\beta} = e_\alpha^{\alpha'} e_\beta^{\beta'} g_{\alpha'\beta'}$, for any $\mathbf{g}_{\alpha\beta}$ (3), we relate the class

of such (inhomogeneous) cosmological solutions, for instance, to the family of metrics of type (43).[18] In explicit form, the solutions can be modelled as explicit diffusions from the h– to v–components of geometrical/physical objects.

Following our recent results on constructing general solutions in Einstein gravity and modifications [17,18,21], we can construct 'non–Killing' solutions depending on all coordinates. Such general classes of sure solutions can be parametrized in the form

$$\mathbf{g} = +g_i(x^k)dx^i \otimes dx^i + \omega^2(x^j,t,y)h_a(x^k,t)\mathbf{e}^a \otimes \mathbf{e}^a,$$
$$\mathbf{e}^3 = dy^3 + w_i(x^k,t)dx^i, \mathbf{e}^4 = dy^4 + n_i(x^k,t)dx^i, \tag{60}$$

for any ω for which

$$\mathbf{e}_k\omega = \partial_k\omega + w_k\omega^* + n_k\partial\omega/\partial y = 0, \tag{61}$$

when (60) with $\omega^2 = 1$ is of type (3).

A class of stochastic solutions with nontrivial sure factor ω satisfying the sure conditions (61) with sure values w_k and n_k but stochastic h_a can be generated similarly to the case of nonholonomic sure Killing configurations. In a more general approach, we can impose the conditions (61) for any stochastic values ω, n_k, w_k and h_a. With respect to N–adapted frames such conditions can be satisfied for certain particular, explicit examples, configurations or in non–explicit forms.

5 Summary and Conclusions

In this work, we have studied the best strategy, and elaborated a geometric method, for generating stochastic exact solutions of Einstein equations in general relativity and modifications.

Our proposal is to follow respectively certain key steps and sub–steps for constructing deterministic and stochastic solutions for nonholonomic gravitational and matter fields interactions:

1. **Nonholonomic splitting & formal integration of Einstein eqs**
   – Let us introduce a nonholonomic $2 + 2$ distribution stating a nonlinear connection (N–connection) structure
     $\mathbf{N}$: $T\mathbf{V} = h\mathbf{V} \oplus v\mathbf{V}$ with associated N–adapted frames, $\mathbf{e}_\alpha = (\mathbf{e}_i, \partial_a)$ and $\mathbf{e}^\beta = (e^i, \mathbf{e}^a)$, on a (pseudo) Riemannian manifold $\mathbf{V}$ enabled with metric structure $\mathbf{g}$, when $\partial_a = \partial/\partial y^a$ and $e^i = dx^i$ for local coordinates $u^\beta = (x^j, y^a)$, or $\mathbf{u} = (\mathbf{x}, \mathbf{y})$.
   – We adapt all geometric constructions to $\mathbf{N}$ and define the canonical distinguished connection(d–connection) $\widehat{\mathbf{D}} = \nabla + \widehat{\mathbf{Z}}$, where ∇ and $\widehat{\mathbf{Z}}$ are respectively the Levi–Civita connection ∇ and the canonical distortion distinguished tensor (d–tensor), all defined in a unique metric compatible form by $\mathbf{g}$. Any N–connection structure $\mathbf{N} = \{N_i^a(\mathbf{u})\}$ states a conventional horizontal–vertical (h-v) decompositions of geometric object, for instance, $\widehat{\mathbf{D}} = h\widehat{D} \oplus v\widehat{D}$ and $\mathbf{g} = hg \oplus vg$, for $hg = \{g_{ij}\}$ and $vg = \{h_{ab}\}$ with respect to N–elongated frames, when $\mathbf{e}_i = \partial_i - N_i^b\partial_b$ and $\mathbf{e}^a = dy^a + N_i^a dx^i$.
   – We write the Einstein equations (36) and (37) for the geometric data $(\mathbf{g}, \mathbf{N}, \widehat{\mathbf{D}})$; the gravitational filed equations are equivalent to those in standard variables $(\mathbf{g}, \nabla)$, see (35), for correspondingly defined energy–momentum and distortion sources.
   – For a general ansatz, we get that the Einstein equations are equivalent to a system of nonlinear PDE (38)–(41) and (42), with a "consequent splitting" of equations which allows us to construct general off–diagonal solutions in sure variables; in local coordinate frames such metrics are parametrized in the form[19]

$$g_{\alpha\beta}(u^\tau) = q \times \begin{vmatrix} g_1 + \omega^2(w_1{}^2h_3 + \omega^2(n_1{}^2h_4) & \omega^2(w_1w_2h_3 + n_1n_2h_4) & \omega^2\ w_1h_3 & \omega^2\ n_1h_4 \\ \omega^2(w_1w_2h_3 + n_1n_2h_4) & g_2 + \omega^2(w_2{}^2h_3 + n_2{}^2h_4) & \omega^2\ w_2h_3 & \omega^2\ n_2h_4 \\ \omega^2\ w_1h_3 & \omega^2\ w_2h_3 & h_3 & 0 \\ \omega^2\ n_1h_4 & \omega^2\ n_2h_4 & 0 & h_4 \end{vmatrix},$$

   where, in this work, $y^a = (y^3 = t, y^4 = y)$ and spacetime signature is chosen $(+, +, -, +)$. The coefficients $g_k(x^i), h_a(x^i, t), w_k(x^i, t), n_k(x^i, t), q(x^i, t)$ and $\omega(x^i, t, y)$ can be defined in explicit form by integrating and/or differentiating some generating functions. For instance, the class of certain solutions (3) is with functional

---

[18] We have to solve certain systems of quadratic algebraic equations and define some $e_\alpha^{\alpha'}(u^\beta)$, choosing a convenient system of coordinates $u^{\alpha'} = u^{\alpha'}(u^\beta)$.

[19] as a matter of principle, any solution of gravitational field equations with certain general matter fields sources can be represented in such a form by corresponding frame and coordinate transforms; we have to involve certain additional physical considerations, suppositions on symmetry of interactions and boundary conditions in order to model realistic gravitational interactions etc.

dependence on a generating function $\phi(x^k, t)$, when $h_a = h_a[x^i, \phi]$ and (following from relations for v–metric) $w_i = w_i[x^i, t, \phi]$ and $n_i = n_i[x^i, t, \phi]$. The type of chosen generating and integration functions (smooth class, sure or stochastic character), prescribed symmetries and topology of interactions, boundary/limit conditions etc distinguish in explicit form a geometric/physically important class of solutions.

2. **Elaborate a nonholonomic stochastic calculus and related general relativistic diffusion theory on V**
   - The theory of stochastic processes and diffusion on curved manifolds, Riemann and Riemann–Cartan spaces [20], and bundle spaces is a well developed direction in modern mathematics.[21]
   - In this work, we have shown how the mathematical formalism for the relativistic stochastic theory and diffusion can be adapted for nonholonomic manifolds with nontrivial N–connection structure. In brief, the constructions are those for (pseudo) Riemannian manifolds but with corresponding generalizations with respect necessary classes of orthonormalized N–adapted frames, modified linear connections and Laplace–Beltramy operators.
   - The geometric data $(\mathbf{g}, \mathbf{N}, \widehat{\mathbf{D}})$ and/or sure general solutions are considered as a fixed nonholonomic background for "non–integrable" rolling of stochastic processes in tangent bundles with h– and v–splitting. So, at this stage the generating/integration functions for the solutions of Einstein equations are taken to be sure functions even fluctuations to nonholonomic Einstein–Langeven systems, induced by quasi–classical quantum fluctuations of matter, can be considered in a self–consistent form as in so–called stochastic gravity [26].

3. **Consider random generating functions and sources for formal solutions of Einstein equations**
   - The key idea of this work is to consider (pseudo) Riemannian metrics with coefficients generalized in a form to include random variables induced by generating functions (for some particular cases, we can consider directly some N–adapted coefficients of v–metric, h_{ab}, and N–connection, N_i^a) defined, for instance, by some gravitational diffusion processes.
   - Transitions from a "sure" nonholonomic (pseudo) Riemannian configuration ($^\circ\mathbf{g}$, $^\circ\mathbf{N}$, $^\circ\widehat{\mathbf{D}}_{|Z=0} \to {^\circ\nabla}$) to a stochastic nonholonomic Einstein spacetime ($^\eta\mathbf{g}$, $^\eta\mathbf{N}$, $^\eta\widehat{\mathbf{D}}_{|Z=0} \to {^\eta\nabla}$) are modeled by metrics of type (3), when $g_{ij} = diag[g_i = \eta_i \, ^\circ g_i]$ and $h_{ab} = diag[h_a = \eta_a \, ^\circ h_a]$ and $N_k^3 = w_i = \eta_i^3 \, ^\circ w_i$ and $N_k^4 = n_i = \eta_i^4 \, ^\circ n_i$ are constructed for some sure and/or stochastic gravitational η–polarizations. The polarizations are defined in such a way that $^\eta\mathbf{g}$ generates a class of exact solutions of the Einstein equations with η_α and η_i^a being related to some stochastic/diffusion processes (as solutions of some generalized and/or nonholonomically constrained Kolmogorov/ Fokker–Planck equations) rolled on a "sure" background ($^\circ\mathbf{g}$, $^\circ\mathbf{N}$, $^\circ\widehat{\mathbf{D}}$). We have to change the usual "sure" integrations into stochastic Stratonovich ones (which is the most convenient for curved spaces), with possible re–definition of results for Itô stochastic integrals. For real physical applications, we have to introduce orthonormalized N–adapted frames related to some certain degrees defined by $^\circ\mathbf{g}$ and sure deformations to $^\eta\mathbf{g}$ but consider that additional deformations are driven by some Wiener processes $\delta\mathcal{W}^{\widehat{a}}$ (for physicists, all necessary concepts on stochastic calculus and diffusion are summarized in Ref. [22]).
   - In explicit form, the nonholonomic transitions between sure and stochastic gravitational configurations are modelled by changing of a "sure" generating function into a random one, $\phi(x^k, t) \to \underline{\phi}(x^k, t) = \phi(x^k, t) + \varpi\widetilde{\phi}(x^k, t)$, see (49). Such mutual evolutions of deterministic and random phases are possible even for "sure" boundary/initial conditions, in classical gravity because of generic nonlinear character of Einstein equations. The stochastic behavior may arise for very small values of parameter ϖ and vanish if $\varpi = 0$. Geometrically, we can impose such nonholonomic splitting into conventional h– and v–subsystems with distinguished sure and stochastic variables, which allows us to split the equations and generate exact solutions.

We emphasize here that it would not be possible to elaborate such, in general, non–perturbative methods of constructing exact solutions in gravity with "sure" and/or "stochastic" variables, and their mixture, if we do not apply corresponding methods from the geometry of nonholonomic manifolds/bundles and N–connection formalism (originally proposed in Finsler and Lagrange geometry and further re–considered in Einstein gravity and modifications).

Let us compare our methods to an alternative approach in stochastic gravity [26]. For various applications in gravity and astrophysics, there are considered semiclassical generalizations of Einstein equation with sources $\langle T_{ab}[g] \rangle'_{ren}$ computed for suitably renormalized expectation value of the stress tensor operator. Such a renormalized value is computed with the scalar field operator satisfying the Klein–Gordon equation on the perturbed metric $g_{ab} + h_{ab}$ and stochastic source ξ_{ab}. We get the so–called Einstein–Langevin equation,

$$^1 G_{ab}[g + h] = \varkappa \langle \, ^1\widehat{T}_{ab}[g + h] \rangle_{ren} + \varkappa \xi_{ab}[g],$$

---

[20] the last ones are with nontrivial torsion; torsion can be also induced nonholonomically on Minkowski and Riemann spaces but there are canonical anholonomic transforms to the Levi–Civita connection with zero torsion

[21] During last two decades the approach was extended for (in general, suppersymmetric) Lagrange–Finsler spaces, and higher order generalizations, and on (pseudo) Riemannian/Lorentz manifolds with various applications of relativistic diffusions in modern cosmology and astrophysics.

where the left superindex means that only terms linear in the metric perturbations are kept (the stochastic source ξ_{ab} is regarded to be of the same order as h_{ab}). The Gaussian stochastic source ξ_{ab} is completely determined by the following correlation functions:

$$\langle\xi_{ab}[g;x]\rangle_\xi = 0, \quad \langle\xi_{ab}[g;x]\xi_{ab}[g;y]\rangle_\xi = N_{abcd}(x,y) = \frac{1}{2}\langle\{\widehat{t}_{ab}[g;x],\widehat{t}_{cd}[g;y]\}\rangle,$$

where $\langle\ldots\rangle_\xi$ denotes the expectation value with respect to the stochastic classical (i.e. non–quantum) source $\xi_{ab}[g]$. The operator $\widehat{t}_{ab}[g]$ is defined as $\widehat{t}_{ab}[g] := \widehat{T}_{ab}[g] - \langle\widehat{T}_{ab}[g]\rangle$ and the bitensor $N_{abcd}(x,y)$, which determines the correlation function of the stochastic source, is computed using the scalar field operator satisfying the Klein–Gordon equation for the background metric g_{ab}. The bitensor $N_{abcd}(x,y)$ is called the noise kernel, it describes the quantum fluctuations of the stress tensor operator and is positive–semidefinite.

Comparing our approach and the stochastic gravity scheme we conclude that we can include that scheme into the anholonomic deformation method considering a subclass of perturbative "fluctuations" of metrics defined by "noise" in energy–momentum tensors. In general, following stochastic nonholonomic geometric constructions we can model generic nonlinear stochastic processes both in vacuum and non–vacuum gravity.

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